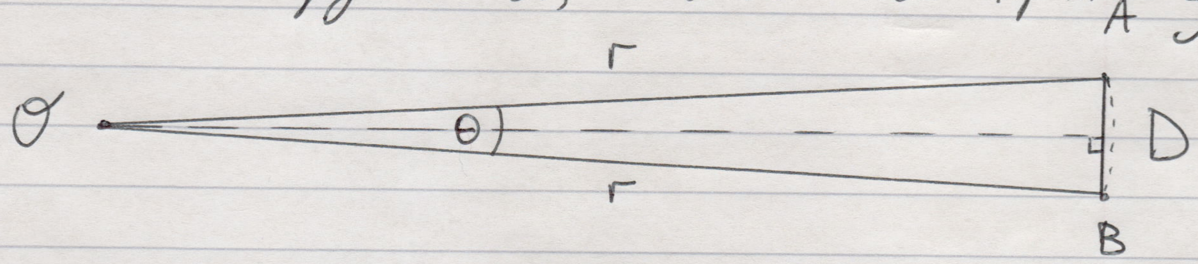
Angular diameter distance

$d_A(z)$

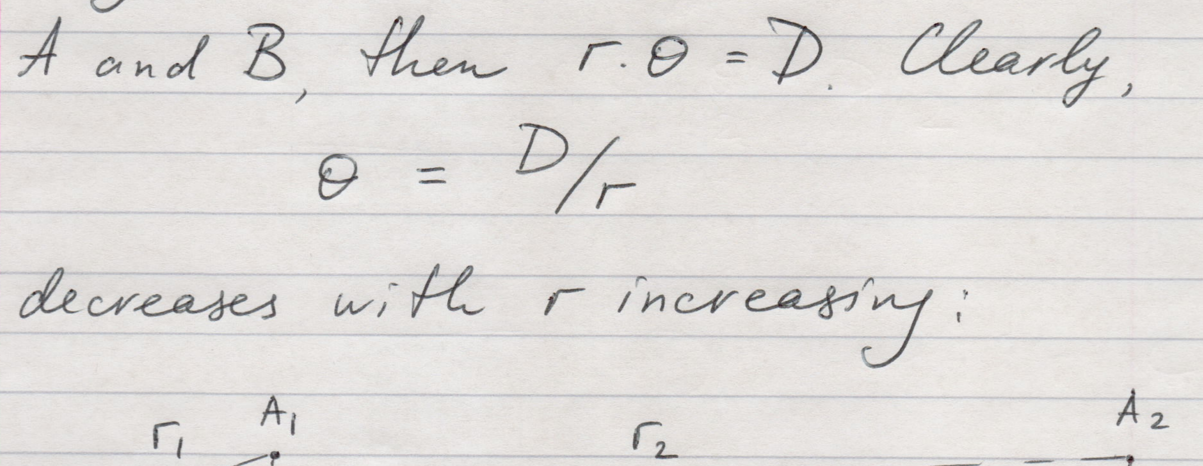
Together with luminosity distance $d_L(z)$, one of the important distances defined in cosmology. First, consider non-expanding space.



In this simple Euclidean construction, we have $r \sin\frac{\theta}{2} = D/2 \Rightarrow r\cdot\theta \approx D$ for $r \gg D$. In fact, we can think of D as being the length of an arc between points A and B, then $r\cdot\theta = D$. Clearly,

$$\theta = D/r$$

decreases with r increasing:



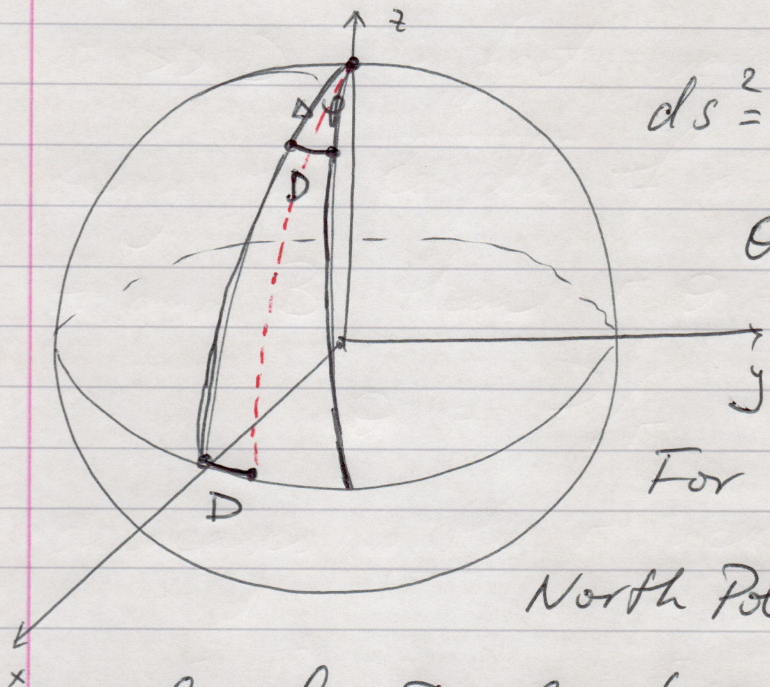
If light is emitted by A and B at $t = t_e$

and received by O at $t = t_0$, then

$$\theta = \frac{D}{c(t_0 - t_e)}$$

If a galaxy of a typical (proper) size D is observed at different increasing r, then θ decreases monotonically with r (or with $t_0 - t_e$).

Now consider curved (but still non-expanding) space, e.g. a sphere:



$$ds^2 = R^2 (d\theta^2 + \sin^2\theta \, d\varphi^2)$$

$\theta = 0$: North pole.

For an observer at the North Pole, an object of arclength D located at $\theta > 0$ has the angular diameter $\Delta\varphi$ computed from

$$D = R \sin\theta \, \Delta\varphi \Rightarrow \Delta\varphi = \frac{D}{R \sin\theta}.$$

$\Delta\varphi$ decreases with θ increasing from 0 to $\pi/2$ (equator) <u>and then increases again</u>.

Now, in the Universe with FRW metric

$$ds^2 = -c^2 dt^2 + R^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right),$$

if a typical galaxy of proper size D emits light at $t = t_e$ (received at $t = t_0$ at $r = 0$), then, with a simple orientation of coordinates,

$$D = R(t_e)\, r\, \Delta\theta,$$

where r is such that a photon emitted at $t = t_e$ at r is received at $t = t_0$ at $r = 0$, i.e. r is related to t_e, t_0 via the geodesic eq. $ds^2 = 0$. For example, with Euclidean 3d metric ($k = 0$),

$dr = -c\, dt/R(t)$, and for $R(t) = (t/t_0)^q$

this is $r = \frac{c t_0}{1-q}\left[1 - \left(\frac{t_e}{t_0}\right)^{1-q}\right]$

Note that $q = 0$ ($R = \text{const}$)
we are back to $r = c(t_0 - t_e)$.

So, one can define $d_A(z)$ as

$$d_A(z) = R(t_e)\, r,$$

with

$$\Delta\Theta = D/d_A .$$

One can express r and $R(t_e)$ via z, $1+z = R_0/R(t_e)$, and $r(z)$ will depend on the model. In the example with $R(t) = (t/t_0)^q$, we have

$$r = \frac{c}{H_0 \alpha}\left[1 - (1+z)^{-\alpha}\right],$$

$\alpha = \frac{1-q}{q}$. Then

$$d_A(z) = \frac{c}{H_0 \alpha (1+z)}\left[1 - (1+z)^{-\alpha}\right].$$

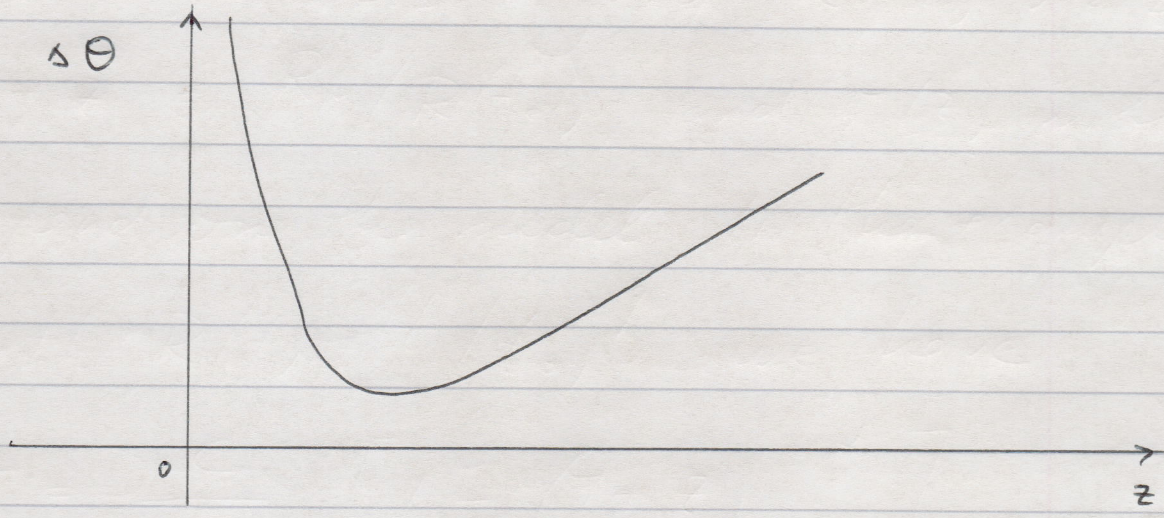
$$\Delta\Theta = \frac{D H_0 \alpha}{c} \frac{(1+z)}{1 - (1+z)^{-\alpha}}$$

In particular, for the matter-dominated

Universe, $q = 2/3$ and $\alpha = 1/2$:

$$\Delta\Theta = \frac{DH_0}{2c} \cdot \frac{1+z}{1-(1+z)^{-1/2}} \; .$$

Note that $\Delta\Theta(z)$ has a min at $z = 1.25$ - this is a <u>typical</u> feature of many (but not all!) cosmological models



We have $\Delta\Theta \sim \frac{DH_0}{2c} \frac{2}{z}$, $z \ll 1$ and

$\Delta\Theta \sim \frac{DH_0}{2c} z$, $z \gg 1$. These limits should be interpreted with some background information in mind. E.g. with $H_0 \sim 70 \frac{km}{s\,Mpc}$ and $D \sim 30$ Kpc, we have $\frac{DH_0}{2c} \sim$ $\sim 3.5 \cdot 10^{-7}$ rad. Most distant galaxies we are able to observe have $z \sim 11$, this gives some understanding of the scales of $\Delta\Theta$ involved.

Formally, remembering that $\Theta = 2 \arcsin D/2r$ and $\arcsin 1 = \pi/2$, we may say that for sufficiently large z, $\Theta \to \pi$, but please keep in mind the realistic numbers above.

Note also:

$$d_L = r R(t_0)(1+z)$$

$$d_A = r R(t_0)/(1+z)$$

$$\Rightarrow d_A = \frac{d_L}{(1+z)^2}.$$

See also Weinberg, „Grav. and cosmology", Ch. 14, Sect. 4 and Hobson, Efstathiou, Lasenby, „General Relativity", Sect. 15.8.