Problem Set 5

Problem 1

1a. $p = 0$ (and $\Lambda = 0$)

Friedmann eq:

$$\frac{\dot{R}^2}{R^2} = -\frac{kc^2}{R^2} \Rightarrow \dot{R}^2 = -kc^2$$

$k = -1 \Rightarrow \dot{R} = 1$ (use $c = 1$ here).

$R(t) = t - t_0$ (let $t_0 = 0$).

$$(*) \quad ds^2 = -dt^2 + t^2\left[\frac{dr^2}{1+r^2} + r^2 d\Omega^2\right]$$

Note: one can check that all components of $R_{\mu\nu\lambda\sigma}$ vanish $\Rightarrow$ must be equiv. to Mink. Alternatively, since $\dot{R}^2 + k = 0$ and $\ddot{R} = 0$, $K = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma} = 0$ (see expression for K above) - non-sing. For spher.-symm. solution in 4d, Birkhoff th. says then that Schwarzschild sol. with $M=0$ - since $K=0$ - is unique - and this is Mink metric ∇

$\Rightarrow$ one can find coord. transf. from (*) to Mink.

1b $\begin{cases} s = s(r,t) & \text{new rad. coord} \\ T = T(r,t) & \text{new time} \end{cases}$

$\Rightarrow$ $s = t\cdot r$ to have $s^2 d\Omega^2$ in the Mink metric. Angular var. are not affected.

1c We need to find

$$\begin{cases} t = t(s,T) \\ r = r(s,T) \end{cases}$$

(and we know that $s = t\cdot r$).

We compute dt, dr, insert into (*) and demand the final form is

$$ds^2 = -dT^2 + ds^2 + s^2 d\Omega^2$$

$$s = rt$$

$$\begin{cases} t = t(s, T) \\ r = r(s, T) \end{cases}$$

$$dt = \left(\frac{\partial t}{\partial s}\right)_T ds + \left(\frac{\partial t}{\partial T}\right)_s dT$$

$$dr = \left(\frac{\partial r}{\partial s}\right)_T ds + \left(\frac{\partial r}{\partial T}\right)_s dT$$

$$\begin{cases} -dt^2 = -\left(\frac{\partial t}{\partial s}\right)_T^2 ds^2 - \left(\frac{\partial t}{\partial T}\right)_s^2 dT^2 - 2\left(\frac{\partial t}{\partial s}\right)_T\left(\frac{\partial t}{\partial T}\right)_s ds\, dT \\ dr^2 = \left(\frac{\partial r}{\partial s}\right)_T^2 ds^2 + \left(\frac{\partial r}{\partial T}\right)_s^2 dT^2 + 2\left(\frac{\partial r}{\partial s}\right)_T\left(\frac{\partial r}{\partial T}\right)_s ds\, dT \end{cases}$$

$$-dt^2 + \frac{t^2}{1+r^2} dr^2 \Rightarrow$$

$$-\left(\frac{\partial t}{\partial T}\right)_s^2 dT^2 + \frac{t^2}{1+r^2}\left(\frac{\partial r}{\partial T}\right)_s^2 dT^2 = -1 \cdot dT^2$$

$$-2\left(\frac{\partial t}{\partial s}\right)_T\left(\frac{\partial t}{\partial T}\right)_s + \frac{2t^2}{1+r^2}\left(\frac{\partial r}{\partial s}\right)_T\left(\frac{\partial r}{\partial T}\right)_s = 0$$

$$\Rightarrow \; -\left(\frac{\partial t}{\partial T}\right)_s^2 + \frac{t^2}{1+r^2}\left(\frac{\partial r}{\partial T}\right)_s^2 = -1$$

Since $r = s/t$, we have

$$\left(\frac{\partial r}{\partial T}\right)_s = -\frac{s}{t^2}\left(\frac{\partial t}{\partial T}\right)_s = -\frac{r}{t}\left(\frac{\partial t}{\partial T}\right)_s .$$

$$\Rightarrow \; -\left(\frac{\partial t}{\partial T}\right)_s^2 + \frac{r^2 \cancel{t}}{(1+r^2)\cancel{t}}\left(\frac{\partial t}{\partial T}\right)_s^2 = -1$$

$$\left(\frac{\partial t}{\partial T}\right)_s^2 \left(\frac{-1}{1+r^2}\right) = -1$$

$$\left(\frac{\partial t}{\partial T}\right)_s^2 = 1+r^2 = 1 + s^2/t^2$$

$$\int \frac{t\,dt}{\sqrt{t^2+s^2}} = \int dT \;\Rightarrow\; T = \sqrt{t^2+s^2}$$

(with integr. const set to zero.)

$$\Rightarrow \boxed{s = rt \qquad T = t\sqrt{1+r^2}}$$

is the transf. we wanted.

1d can be checked with a bit of pain.

Problem 2

Friedmann eq.

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{c^2\Lambda}{3}$$

- flat Univ. $\Rightarrow k = 0$
- Implicit assumpt. : $\Lambda = 0$
- $\rho = \rho_R + \rho_M$

We know: $\rho_R = \rho_{R,0}/R^4$

$\rho_M = \rho_{M,0}/R^3$

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\left(\frac{\rho_{R,0}}{R^4} + \frac{\rho_{M,0}}{R^3}\right)$$

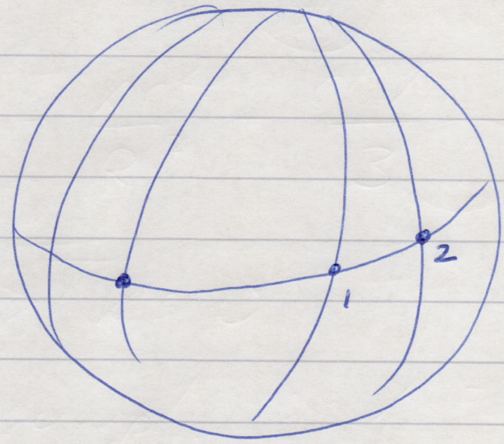
$$\Rightarrow \dot{R} = \sqrt{\frac{8\pi G}{3}}\left(\frac{\rho_{R,0}}{R^2} + \frac{\rho_{M,0}}{R}\right)^{1/2}$$

$$\Rightarrow \int_0^R \frac{R\,dR}{(I+R)^{1/2}} = \int_0^t \left(\frac{8\pi G\rho_{M,0}}{3}\right)^{1/2} dt$$

$I \equiv \rho_{R,0}/\rho_{M,0}$

$\Rightarrow$ gives result with $\Omega_{M,0}^2 = 8\pi G\rho_{M,0}/3H_0^2$

- Comoving vs phys. dist.
  coord.



- Cosmological redshift

$$z = \frac{\lambda_{obs} - \lambda_{em}}{\lambda_{em}} = \frac{R(t_{obs})}{R(t_{em})} - 1$$

$$\Rightarrow \quad 1 + z = \frac{1}{R(t)} \qquad \text{if } t_{obs} = t_0$$

(„now")

$R(t_0) = 1.$

- Luminosity distance $d_L(z)$

$$\Phi_{obs} = \frac{L_{em}}{4\pi d_L^2}$$

$$d_L = \frac{c}{H_0}\left[z + \frac{1}{2}(1 - q_0)\, z^2 + \ldots \right]$$

$q = - R\ddot{R} / \dot{R}^2$ deceleration parameter

- Thermal history of the Universe

$$T(t)\, R(t) = \text{const}$$

- Horizons

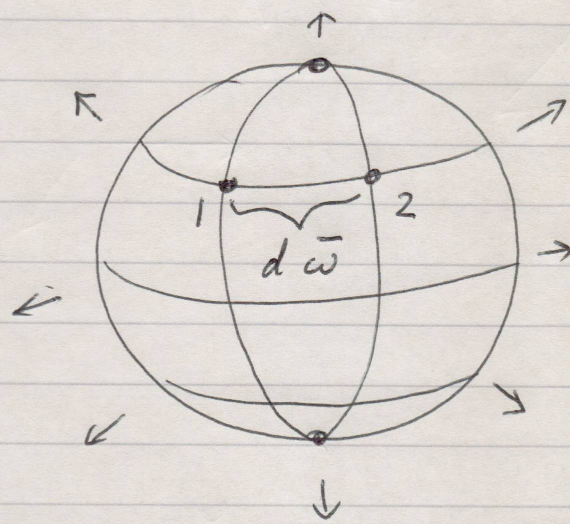
- Age of the Universe

Problem 3

$$ds^2 = -c^2 dt^2 + R^2(t)\left(dr^2 + r^2 d\Omega^2\right)$$

Proper (radial) distance: $d\varpi = R(t)\,dr$

comoving distance



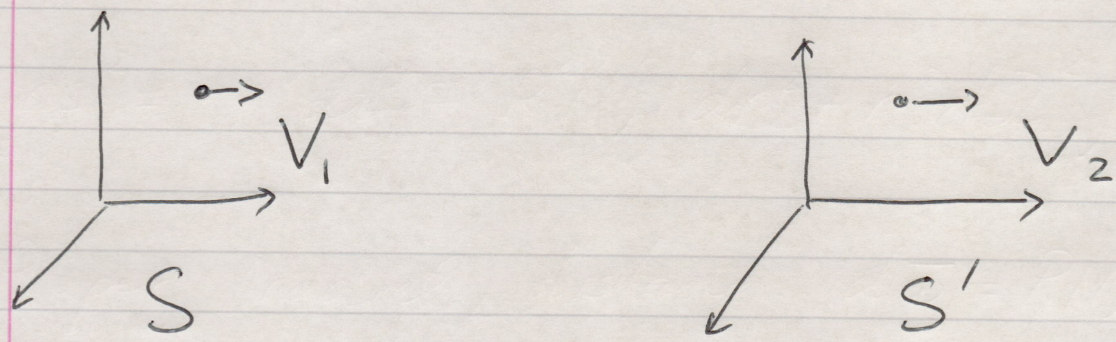
Comoving dist: $2-1=1$

Phys. dist: $\Delta\varpi = R\,\Delta r$

On the inflating balloon, points 1 and 2 (coordinate labels) with comoving distance $\Delta r = 2-1=1$ recede from each other with velocity $V = \dot{\varpi} = \dot{R}\,\Delta r = \frac{\dot{R}}{R} R\,\Delta r$,

i.e. $V = \frac{\dot{R}}{R}\Delta\varpi$. Since $\dot{R}/R$ does not change much on time-scales associated with distance $\Delta\varpi$, we can view V as constant and use special rel.

The standard SR addition of vel.



$$V_2 = \frac{V_1 - V}{1 - \frac{V_1 V}{c^2}} = \frac{V_1 - \frac{\dot{R}}{R}\Delta\bar{\omega}}{1 - \frac{V_1}{c^2}\frac{\dot{R}}{R}\Delta\bar{\omega}}$$

For $\frac{\dot{R}\Delta\bar{\omega}}{Rc} \ll 1$, this is

$$V_2 = \left(V_1 - \frac{\dot{R}}{R}\Delta\bar{\omega}\right)\left(1 + \frac{V_1}{c^2}\frac{\dot{R}}{R}\Delta\bar{\omega} + \cdots\right) =$$

$$= V_1 - \frac{\dot{R}}{R}\Delta\bar{\omega}\left(1 - \frac{V_1^2}{c^2}\right) + \cdots$$

$$\Rightarrow \boxed{V_2 = V_1 - \frac{\dot{R}}{R}\Delta\bar{\omega}\left(1 - \frac{V_1^2}{c^2}\right)}$$

Taking the limit $\Delta\bar{\omega} \to 0$, $V_2 \to V_1 \to V = \frac{d\bar{\omega}}{dt}$,

and $\quad \frac{dV}{dt} = -\frac{\dot{R}}{R} V \left(1 - V^2/c^2\right)$

Thus, $$\frac{\dot{V}}{V(1-V^2/c^2)} = -\frac{\dot{R}}{R}$$

So, $$\int \frac{dV}{V(1-V^2/c^2)} = -\int \frac{dR}{R}$$

Since $$\frac{1}{V(1-V^2/c^2)} = \frac{1}{V} + \frac{1}{2(c-V)} - \frac{1}{2}\frac{1}{(c+V)},$$

we have

$$\ln V - \frac{1}{2}\ln(1-V/c) - \frac{1}{2}\ln(1+V/c) =$$

$$= -\ln R + \text{const}$$

Now, at $t=t_0$ we have $V=V_0$ and $R(t_0)=1$, so $\text{const} = \dfrac{V_0}{\sqrt{1-V_0^2/c^2}} =$

$= \gamma(V_0)V_0 = U_0$ (<u>spatial</u> component of 4-vel. $U^\mu = (\gamma c, \gamma\vec{v})$).

$$\Rightarrow \boxed{\frac{V}{\sqrt{1-V^2/c^2}} = \frac{U_0}{R(t)}}.$$

3b) Since $\bar{p} = \gamma m \bar{v}$, the result in 3a) implies $pR = \text{const}$.

- Adiabatic expansion of a gas of ultra-rel. particles ($\varepsilon = \sqrt{p^2c^2+m^2c^4} \approx pc$) at temperature T: $TR \sim \text{const}$

  But $T \sim \varepsilon \sim p$ in this case, so

  $pR = \text{const}$.

- Non-rel. gas: $T\rho^{-2/3} \sim \text{const}$

  In this case, $\varepsilon \sim p^2/2m$, so

  $T \sim p^2$. Number density $\rho \sim \frac{N}{R^3}$

  $\Rightarrow R \sim \rho^{-1/3} \Rightarrow T \sim \rho^{2/3} \sim \frac{1}{R^2} \sim p^2$

  $\Rightarrow pR \sim \text{const}$.

3c) $d\bar{\omega}/dt = V(R)$

From 3a), $V = \dfrac{c}{\sqrt{1 + c^2R^2/V_0^2}}$

Since $R(t) = (t/t_\circ)^{2/3}$, we have

$$\dot{R} = \frac{2}{3t_\circ}\left(t/t_\circ\right)^{-1/3}, \quad \dot{R}(t_\circ) = \frac{2}{3t_\circ} = H_\circ,$$

since $R(t_\circ) = 1$, $\dot{R}(t_\circ)/R(t_\circ) \equiv H_\circ$.

$$\Rightarrow R\dot{R} = H_\circ R^{1/2} \Rightarrow dt = \frac{R^{1/2}dR}{H_\circ}.$$

So, $$d\bar{\omega} = R\,dr = \frac{c\,dt}{\sqrt{1+c^2R^2/U_\circ^2}}$$

$$\left(d\bar{\omega} = d(Rr) = dR \cdot r + R\,dr \approx R\,dr \right)$$

$$\Rightarrow r(R) = \frac{c}{H_\circ}\int_0^R \frac{dR}{R^{1/2}\sqrt{1+c^2R^2/U_\circ^2}}$$

or $$\boxed{r(R) = \frac{c}{H_\circ}\int_0^R \frac{dx}{\sqrt{x + c^2x^3/U_\circ^2}}}$$

This can be rewritten as (with a change of var. $x = \alpha y$):

$$r(R) = \frac{c}{H_0}\sqrt{\alpha}\int_0^{R/\alpha} \frac{dy}{\sqrt{y + y^3}},$$

where $\alpha = U_0/c$. In particular,

$$r_{max} = \frac{\sqrt{U_0 c}}{H_0}\int_0^{\infty} \frac{dy}{\sqrt{y + y^3}} \simeq 3.708\,\frac{\sqrt{U_0 c}}{H_0}.$$

Problem 4

Friedmann eq

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3}\rho - \frac{kc^2}{R^2} + \frac{\Lambda c^2}{3}$$

In our case, $\Lambda = 0$, $\rho = \rho_m$.

Note: $\rho_m R^3 = \rho_{0,m} R_0^3 = \rho_{0,m}$

$$\Rightarrow \rho_m = \rho_{0,m} / R^3$$

$$\frac{\dot{R}^2}{R^2} = H^2 = \frac{H_0^2}{R^2}\left(\frac{8\pi G \rho_{0m}}{3H_0^2}\frac{1}{R} - \frac{kc^2}{H_0^2} \right)$$

Introduce $\Omega_{m0} \equiv 8\pi G \rho_{0m} / 3H_0^2$

Then: $\Omega_{m0} - \frac{kc^2}{H_0^2} = 1$

We can write Friedmann eq as

$$\frac{\dot{R}^2}{R^2} = H_0^2 \left(\frac{\Omega_{m0}}{R^3} + \frac{1-\Omega_{m0}}{R^2} \right)$$

At $R = R_{max}$, $\dot{R} = 0 \Rightarrow R_{max} = \frac{-\Omega_{m0}}{1-\Omega_{m0}}$.

$$R_{max} = \frac{1}{1 - \Omega_{M0}^{-1}} .$$

When $R = R_{max}$, the matter density is

$$\rho_{M,max} = \rho_{0,M} \left(1 - \Omega_{M0}^{-1}\right)^3 .$$

Time can be found by integrating

$$\dot{R}^2 = H_0^2 \left(\frac{\Omega_{M0}}{R} + 1 - \Omega_{M0} \right)$$

$$\Rightarrow \int_0^T dt = \int_0^{R_{max}} \frac{\sqrt{R}\, dR}{H_0 \left(\Omega_{M0} - (\Omega_{M0} - 1) R \right)^{1/2}}$$

(Expansion from 0 to R_{max} and collapse from R_{max} to 0 takes the same time.)

$$\boxed{T = \frac{\pi\, \Omega_{M0}}{2H_0 \left(\Omega_{M0} - 1\right)^{3/2}}}$$

$$T = \frac{\pi}{2H_0} \Omega_{M0}^{-1/2} \left(1 - \Omega_{M0}^{-1}\right)^{-3/2} = \frac{\pi}{2H_0} \Omega_{M0}^{-1/2} \left(\frac{\rho_{0,M}}{\rho_{M,max}} \right)^{1/2}$$

Since $\Omega_{M,0} = 8\pi G \rho_{0M} / 3H_0^2$, we

have:

$$T = \left(\frac{3\pi}{32 G \rho_{M,max}} \right)^{1/2}.$$

We now consider particle falling from rest at $r = r_0$ to $r = r_s$ in Schwarzschild geometry with $M = \frac{4}{3}\pi r_0^3 \rho_{max}$.

Radial geodesic obeys $\frac{\dot{r}^2}{2} + V_{eff} = \text{const.}$

Radial infall $\Rightarrow J = 0$, so it is just

$$\frac{\dot{r}^2}{2} - \frac{GM}{r} = \text{const} = -\frac{GM}{r_0} \quad ; \quad \dot{r} \equiv \frac{dr}{d\tau}.$$

(see e.g. eq 353 of Lecture Notes)

Integrating, we get (the sign reflects dir. of velocity)

$$\frac{1}{\sqrt{2GM}} \int_{r_0}^{r_s} \frac{\sqrt{r}\, dr}{\sqrt{1 - r/r_0}} = -\int d\tau$$

With $x = r/r_0$, this is

$$\frac{r_0^{3/2}}{\sqrt{2GM}} \int_{r_s/r_0}^{1} \frac{\sqrt{x}\, dx}{\sqrt{1-x}} = \tau$$

We can simplify life by setting $r_s/r_0 \to 0$

Then

$$\frac{r_0^{3/2}}{\sqrt{2GM}} \int_0^1 \frac{\sqrt{x}}{\sqrt{1-x}}\, dx = \frac{\pi}{2} \frac{r_0^{3/2}}{\sqrt{2GM}} = \tau$$

$$\Rightarrow \tau = \sqrt{\frac{3\pi}{32 G \rho_{max}}}$$ Curious...

5. a) FRW eq is

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{R^2}$$

$k > 0$ for closed Universe

$$H^2(t) \equiv \dot{R}^2/R^2.$$

$$\frac{kc^2}{R^2} = \frac{8\pi G\rho}{3} - H^2 = H^2\left(\frac{8\pi G\rho}{3H^2} - 1\right)$$

Evaluating at $t = t_0$ (now), we have

$$\frac{kc^2}{R_0^2} = H_0^2\left(\frac{8\pi G\rho_0}{3H_0^2} - 1\right) = H_0^2\left(\Omega_{M,0} - 1\right)$$

With $R_0 = 1$ and $k = 1/L^2$, this is

$$H_0^2\left(\Omega_{M,0} - 1\right) = c^2/L^2,$$ where

$$\Omega_{M,0} \equiv 8\pi G\rho_0/3H_0^2 > 1$$

b) A photon travels along the geodesic $ds^2 = 0$, i.e. $-c^2dt^2 + \frac{R^2dr^2}{1-kr^2} = 0$ (for radial geodesics). Here $k = 1/L^2 > 0$, where L is the radius of the (closed)

Universe. This can be integrated

$$c\int_0^t \frac{dt}{R} = \int_0^r \frac{dr}{\sqrt{1-r^2/L^2}}$$

With e.g. $r/L = \sin x$, the r.h.s. gives

$$\int_0^r \frac{dr}{\sqrt{1-r^2/L^2}} = L\int_0^{\arcsin r/L} dx = L \arcsin r/L.$$

To deal with the l.h.s., we use Friedmann eq. for matter-dominated Universe

$(R^3\rho = \text{const} \Rightarrow \rho = \rho_{M,0}/R^3)$

$$\dot{R}^2 = \frac{8\pi G\rho_{M,0}}{3R} - kc^2 \qquad (*)$$

Note that with $R_0 = 1$, we have

$$H_0^2 \equiv \frac{\dot{R}^2}{R^2}(t=t_0) = \frac{8\pi G\rho_{M,0}}{3} - kc^2$$

or $1 = \Omega_{M,0} - \frac{kc^2}{H_0^2}$, where

$$\Omega_{M,0} = \frac{8\pi G \rho_{M,0}}{3 H_0^2}.$$

Then eq (*) can be re-written as

$$\dot{R}^2 = H_0^2 \left(\frac{\Omega_{M,0}}{R} + 1 - \Omega_{M,0} \right)$$

or

$$\frac{dR}{\sqrt{\Omega_{M,0}/R + 1 - \Omega_{M,0}}} = H_0 \, dt$$

This can be integrated directly to give $R(t)$ or else expressed in parametric form as in the lecture notes (§ 9.5):

$$R = \frac{1 - \cos\eta}{2(1 - \Omega_{M0}^{-1})} \quad \Rightarrow$$

$$H_0 t = \frac{\eta - \sin\eta}{2\sqrt{\Omega_{M,0}(1 - \Omega_{M,0})^3}}. \quad (**)$$

We use this parametrization to rewrite

$c \int_0^t \frac{dt}{R}$ as (use ** for $dt \to d\eta$):

$$c\int_0^t \frac{dt}{R} = c\int_0^{\eta_*} \frac{d\eta \; 2\left(1-\Omega_{M,0}^{-1}\right)}{2H_0\sqrt{\Omega_{M,0}\left(1-\Omega_{M,0}^{-1}\right)^3}} =$$

$$= \frac{c}{H_0\sqrt{\Omega_{M,0}-1}}\int_0^{\eta_*} d\eta = \frac{c\,\eta_*}{H_0\sqrt{\Omega_{M,0}-1}},$$

where η_* corresponds to t as in eq $**$. We use η instead of η_* in the following. Also, recall that

$$\Omega_{M,0}-1 = \frac{kc^2}{H_0^2} = c^2/H_0^2L^2$$

So, the photon's geodesic is:

$$\frac{\cancel{c}\,\eta\,\cancel{H_0}L}{\cancel{H_0}\,\cancel{c}} = L\arcsin r/L$$

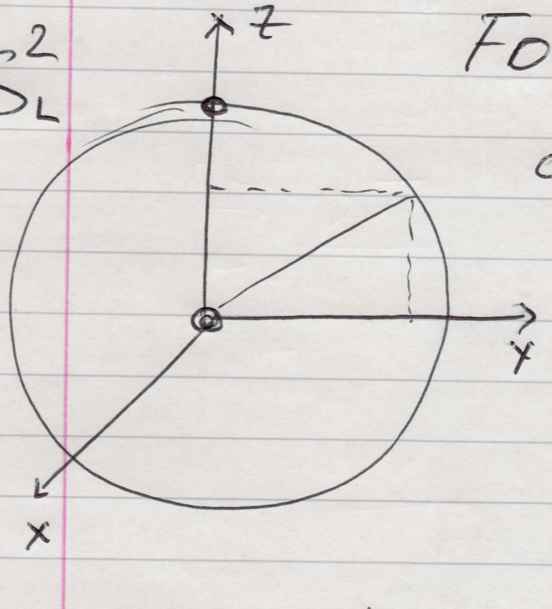
i.e. $\eta = \arcsin r/L$.

Or $\sin\eta = r/L$. A photon travels from $r=0$ to $r=L$ and back to

$r = 0$ during „time" $\Delta\eta = \pi$.

We should be careful in remembering what r is when $K \neq 0$. This can be illustrated by the 2d spherical geom[etry] of the balloon embedded in 3d spa[ce] with the metric $ds^2_{3d} = dz^2 + dr^2 + r^2 d\varphi$[^2] in cylindrical coordinates.

S^2_L



For 2d sphere, we have a constraint: $z^2 + r^2 = L^2$

$$\Rightarrow dz = -r\,dr/z$$

$$\Rightarrow ds^2_{S^2} = \frac{dr^2}{1 - r^2/L^2} + r^2 d\varphi^2.$$

$r \in [0, L)$, but $z = \pm\sqrt{L^2 - r^2}$, with $\pm$ corresponding to upper/lower hemisph[ere]. Ignoring expansion of the balloon her[e], the „photon" geodesics are

$$\pm c\,dt = dr \big/ \sqrt{1 - r^2/L^2}, \text{ i.e.}$$

$r = \pm L \sin \frac{c\Delta t}{L}$. Starting from e.g. North Pole at $\Delta t = 0$, the photon reaches $r = L$ when $\frac{c\Delta t}{L} = \frac{\pi}{2}$ and the South Pole when $\frac{c\Delta t}{L} = \pi$. Note that the South Pole corresponds to $r = 0$. The full trip to North Pole takes $\frac{c\Delta t}{L} = 2\pi$.

In our case, $r = \pm L \sin \eta$ and also $R(\eta) = \frac{1 - \cos\eta}{2(1 - \Omega_{M,0}^{-1})}$. The full cycle of the Universe from $R = 0$ at $\eta = 0$ to max R at $\eta = \pi$ and back to $R = 0$ at $\eta = 2\pi$ takes $\Delta\eta = 2\pi$. A photon reaches the „South Pole" (max comoving distance on the surface of the balloon but corresp. to $r = 0$) at $\eta = \pi$ and comes back to North Pole at $\Delta\eta = 2\pi$ $\Rightarrow$ makes 1 trip around Universe.