OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

SYMMETRY AND RELATIVITY

TUTORIAL VII

Forces and fields

Problem Set 4

(Part B: problems 4-8)

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Reminder:

Maxwell's equations in $3d$ notations

$$\begin{aligned}
&\nabla \cdot \mathbf{B} = 0\,, \\
&\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \\
&\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,, \\
&\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

can be written in covariant form as follows (in Minkowski signature $(- + ++)$):

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu\,, \tag{1}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0\,, \tag{2}$$

where $J^\nu = (\rho c, \rho \mathbf{v})$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $A^\mu = (\phi/c, \mathbf{A})$, i.e.

$$F^{\mu\nu} = F = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.$$

Note that $det F = (\mathbf{E} \cdot \mathbf{B})^2 / c^2$.

Also, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$, i.e.

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0\,. \end{pmatrix}.$$

Maxwell's equations (with a given J^μ) are *linear* PDEs: their generic solution is thus known (for example, it can be obtained via Fourier transform and Green's functions method of solving linear PDEs). One should keep in mind, however, that J^μ is produced by charged particles whose equations of motion (containing Lorentz force) are coupled to electromagnetic fields, including the ones generated by the particles. One simple consideratio is that an accelerated charge radiates and therefore loses energy, so its law of motion is inevitably affected. The issue is not very simple, although in most model situations it can be avoided by making suitable approximations. A discussion of these issues can be found e.g. in Chapter 16 of [1] (expanded in the 3rd - 1998 - or later editions).

Problem 5

The electromagnetic field of a charge in an arbitrary state of motion is given by

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0\kappa^3}\left(\frac{\hat{\mathbf{n}} - \hat{v}/c}{\gamma^2 r^2} + \frac{\hat{\mathbf{n}} \wedge [(\hat{\mathbf{n}} - \hat{v}/c) \wedge \mathbf{a}]}{c^2 r}\right),$$

where $\hat{\mathbf{n}} = \mathbf{r}/r$ *and* $\kappa = 1 - v_r/c = 1 - \hat{\mathbf{n}} \cdot \mathbf{v}/c$*, and*

$$\mathbf{B} = \hat{\mathbf{n}} \wedge \mathbf{E}/c\,,$$

where $\mathbf{r}$ *is the vector from the source point to the field point, and* $\mathbf{v}$ *and* $\mathbf{a}$ *are the velocity and acceleration of the charge at the source event. Without detailed derivation, outline briefly how this result may be obtained. How is the source event identified?*

A charged particle moves along the x *axis with constant proper acceleration ("hyperbolic motion"), its worldline being given by*

$$x^2 - t^2 = \alpha^2$$

in units where $c = 1$*. Find the electric field at* $t = 0$ *at points in the plane* $x = \alpha$*, as follows:*

(i) Consider the field event $(t, x, y, z) = (0, \alpha, y, 0)$*. Show that the source event is at*

$$x_s = \alpha + \frac{y^2}{2\alpha}\,.$$

(ii) Show that the velocity and acceleration at the source event are

$$v_s = -\frac{\sqrt{x_s^2 - \alpha^2}}{x_s}\,,$$

$$a_s = \frac{\alpha^2}{x_s^3}\,.$$

(iii) Consider the case $\alpha = 1$*, and the field point* $y = 2$*. Write down the values of* x_s*,* v_s*, and* a_s*. Draw on a diagram the field point, the source point, and the location of the charge at* $t = 0$*. Mark at the field point on the diagram the directions of the vectors* $\hat{\mathbf{n}}$*,* $\mathbf{v}$*,* $\mathbf{a}$*, and* $\hat{\mathbf{n}} \wedge (\hat{\mathbf{n}} \wedge \mathbf{a})$*. Hence, by applying the formula above, establish the direction of the electric field at* $(t, x, y, z) = (0, 1, 2, 0)$*.*

(iv) If two such particles travel abreast, undergoing the same motion, but fixed to a rod perpendicular to the x *axis such that their separation is constant, comment on the forces they exert on one another.*

Solution:

We shall change notations somewhat, to be consistent with those used in Problem 4:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0\kappa^3}\left(\frac{\hat{\mathbf{n}} - \hat{v}_0/c}{\gamma^2 R^2} + \frac{\hat{\mathbf{n}} \wedge [(\hat{\mathbf{n}} - \hat{v}_0/c) \wedge \mathbf{a}_0]}{c^2 R}\right), \tag{3}$$

$$\mathbf{B} = \hat{\mathbf{n}} \wedge \mathbf{E}/c\,, \tag{4}$$

where $\hat{\mathbf{n}} = \mathbf{R}/R$ and $\kappa = 1 - v_r/c = 1 - \hat{\mathbf{n}} \cdot \mathbf{v}_0/c$, $\mathbf{R}(\tau_r) = \mathbf{r} - \mathbf{r}_0(\tau_R)$, $R = |\mathbf{R}|$, and $\mathbf{r}_0$, $\mathbf{v}_0$, $\mathbf{a}_0$ are all computed at time τ_R, as discussed in Problem 4. These expressions can be derived from the Lienard-Wiechert potentials

$$\phi(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{R - \mathbf{R}\mathbf{v}_0/c}\,, \tag{5}$$

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 q}{4\pi} \frac{\mathbf{v}_0}{R - \mathbf{R}\mathbf{v}_0/c} \tag{6}$$

using the standard formula

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}$$

and remembering that $\tau_R = \tau_R(t, \mathbf{r})$, so

$$\frac{\partial}{\partial t} = \frac{\partial \tau_R}{\partial t} \frac{\partial}{\partial \tau_R}\,.$$

Since

$$\tau_R = t - \frac{|\mathbf{r} - \mathbf{r}_0(\tau_R)|}{c} \tag{7}$$

we have

$$\frac{\partial \tau_R}{\partial t} = 1 + \frac{\mathbf{v}_0 \cdot (\mathbf{r} - \mathbf{r}_0)}{c|(\mathbf{r} - \mathbf{r}_0)|} \frac{\partial \tau_R}{\partial t}\,, \tag{8}$$

i.e.

$$\frac{\partial \tau_R}{\partial t} = \frac{1}{1 - \frac{\mathbf{v}_0 \cdot (\mathbf{r} - \mathbf{r}_0)}{c|(\mathbf{r} - \mathbf{r}_0)|}}\,. \tag{9}$$

This somewhat messy calculation eventually gives the result for $\mathbf{E}$ and $\mathbf{B}$.

The "source event" is identified by solving Eq. (7) for τ_R: this gives $\tau_R = \tau_R(t, x, y, z)$, and then $\mathbf{r}_0(\tau_R)$ gives the spatial coordinates for the source at τ_R for any point of observation P specified by t, x, y, z.

We now consider hyperbolic motion $x = x_0(t) = \sqrt{\alpha^2 + t^2}$.

i) Consider the point of observation $P = (t, x, y, z) = (0, \alpha, y, 0)$ for any y. From Eq. (7) we get for τ_R:

$$\tau_R = -\sqrt{\alpha^2 + y^2 - 2\alpha x_0(\tau_R) + \tau_R^2 + \alpha^2}\,. \tag{10}$$

Indeed, since $t = 0$, we have $|\mathbf{r} - \mathbf{r}_0(\tau_R)|^2 = (x - x_0(\tau_R))^2 + y^2 + z^2$, where $x_0(\tau_R) = \tau_R^2 + \alpha^2$. For $z = 0$, Eq. (10) gives

$$x_0(\tau_R) = \alpha + \frac{y^2}{2\alpha}\,.$$

Note that $\tau_R < 0$: the particle is coming from $x = \infty$ decelerating towards $x = \alpha$, then goes back to $x = \infty$ for $t > 0$.

ii) Velocity is found from $x^2 - t^2 = \alpha^2$: we have $2x\dot{x} - 2t = 0$ and thus $\dot{x} = t/x$. In particular,

$$v_0(\tau_R) = \tau_R/x_0(\tau_R) = -\frac{\sqrt{x_0^2(\tau_R) - \alpha^2}}{x_0(\tau_R)}\,.$$

Acceleration can be computed by taking another derivative of $2x\dot{x} - 2t = 0$ which gives $\dot{x}^2 + x\ddot{x} - 1 = 0$. From this, we find

$$\ddot{x} = \frac{1 - \dot{x}^2}{x} = \frac{x^2 - t^2}{x^3} = \frac{\alpha^2}{x^3}\,.$$

In summary:

$$v_0(\tau_R) = \tau_R/x_0(\tau_R) = -\frac{\sqrt{x_0^2(\tau_R) - \alpha^2}}{x_0(\tau_R)} \tag{11}$$

$$a_0(\tau_R) = \frac{\alpha^2}{x_0^3(\tau_R)}\,. \tag{12}$$

iii) Now consider $\alpha = 1$, $y = 2$. Then

$$x_0(\tau_R) = \alpha + \frac{y^2}{2\alpha} = 3\,, \tag{13}$$

$$v_0(\tau_R) = \tau_R/x_0(\tau_R) = -\frac{\sqrt{x_0^2(\tau_R) - \alpha^2}}{x_0(\tau_R)} = -\frac{2\sqrt{2}}{3}\,, \tag{14}$$

$$a_0(\tau_R) = \frac{\alpha^2}{x_0^3(\tau_R)} = \frac{1}{27}\,. \tag{15}$$

We now compute the ingredients of the expression for $\mathbf{E}$. First,

$$\mathbf{r} - \mathbf{r}_0(\tau_R) = (x - x_0(\tau_R), y, z) = (-2, 2, 0)\,.$$

The magnitude of this vector is $|\mathbf{r} - \mathbf{r}_0(\tau_R)| = 2\sqrt{2}$. Then

$$\mathbf{n} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)\,.$$

Other ingredients are $\gamma^2 = 1/(1 - v_0^2) = 9$, $\kappa = 1 - \mathbf{n}\cdot\mathbf{v}_0 = 1 - 2/3 = 1/3$. We also have

$$\mathbf{n} - \mathbf{v}_0 = \left(-\frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{3}, \frac{1}{\sqrt{2}}, 0\right)$$

and

$$\frac{\mathbf{n} - \mathbf{v}_0}{\gamma^2 R} = \frac{1}{18}\left(\frac{1}{6}, \frac{1}{2}, 0\right) .$$

Finally,

$$\mathbf{n} \times [(\mathbf{n} - \mathbf{v}_0) \times \mathbf{a}] = (\mathbf{n} - \mathbf{v}_0)(\mathbf{n} \cdot \mathbf{a}) - \mathbf{a}(1 - \mathbf{n}\mathbf{v}_0) = -\frac{1}{27 \cdot 2}(1, 1, 0) .$$

Assembling all ingredients, we find the electric field:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{27}{2\sqrt{2}}\left(-\frac{1}{108}, \frac{1}{108}, 0\right) = \frac{q}{4\pi\epsilon_0}\frac{\sqrt{2}}{16}(-1, 1, 0) .$$

We conclude that the direction of $\mathbf{E}$ at the point $(1, 2, 0)$ at $t = 0$ is along vector $\mathbf{n}$.

iv) For a particle of the same charge, $\mathbf{F} = q\mathbf{E}$ at point P (note that $\mathbf{B} = 0$ at P, since $\mathbf{E}$ and $\mathbf{n}$ are parallel, and $\mathbf{B} = \mathbf{n} \times \mathbf{E}/c$). There is an obvious repulsion along OY (balanced by the rod), but there is also a negative x component of the force (in the direction opposite to the direction of $\mathbf{a}$) - a “radiative friction” of sorts.

Problem 6

The far field due to an elementary wire segment dz carrying oscillating current I is given by

$$dE = \frac{I \sin\theta}{2\epsilon_0 c r} \frac{dz}{\lambda} \cos(kr - \omega t) .$$

Compare and contrast the case of a short antenna and the half-wave dipole antenna. Roughly estimate E in the far field for each case by proposing a suitable model for the distribution of current $I(z)$ in the antenna. What happens (qualitatively) for still longer antennae?

Solution:

The details of the antenna story can be found, for example, in Section 8.2.5 of the Steane's book [2]. Essentially, we are considering a solution to Maxwell's equations for an oscillating dipole. This topic has numerous applications (see Fig. 1).

FIG. 1: Long-range submarine communication centre "Vilejka" of the USSR Navy. The height of antennas: 305 m. Radiated power: 1 MW. Distance: 10 000 km, water penetration depth: 60-70 m.

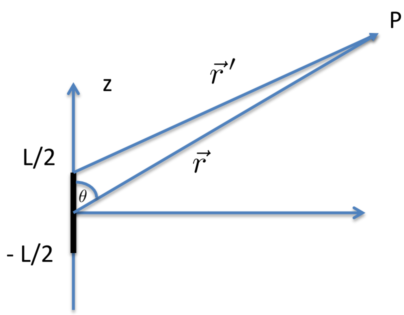

FIG. 2:

We are considering distances much larger than the size of the antenna: $r \equiv |\mathbf{r}| \ll L$ (see Fig. 2).

From Fig. 2, we can see that $\mathbf{r}'^2 = z^2 + \mathbf{r}^2 - 2z|\mathbf{r}|\cos\theta$. We can expand

$$r'(z) = r\sqrt{1 - \frac{2z}{r}\cos\theta + \frac{z^2}{r^2}} \approx r\left(1 - \frac{z}{r}\cos\theta + \cdots\right) ,$$

i.e. $r' \approx r - z\cos\theta$ (in the far field). We need to compute

$$E(r) = \frac{\sin\theta}{2\epsilon_0 \lambda c} \int \frac{I(z)\cos\left(kr'(z) - \omega t\right)}{r'(z)} dz \,. \tag{16}$$

We can use $\exp i(kr'(z) - \omega t)$ and then take the real part. Consider the following two examples of the current distribution $I(z)$:

i) A short antenna: the current distribution model is

$$I(z) = I_0 \left(1 - \frac{2|z|}{L}\right) . \tag{17}$$

Integrating (16) over z and using $r' \approx r$ in the denominator of (16) but not in the phase $\exp\left(kr'(z) - \omega t\right)$, we find

$$E(r) = \frac{\sin\theta}{2\epsilon_0 \lambda c} \frac{e^{i(kr - \omega t)}}{r} \frac{8I_0 \sin^2(\frac{kL\cos\theta}{4})}{k^2 L \cos^2\theta} . \tag{18}$$

For $kL \ll 1$ (i.e. for $L \ll \lambda$, since $k = 2\pi/\lambda$), this simplifies to

$$E(r) = \frac{I_0 L \sin\theta}{4\epsilon_0 \lambda c} \frac{e^{ikr - i\omega t}}{r} . \tag{19}$$

The radiated power is $P \sim E^2 \sim I_0^2 L^2/\lambda^2$ for short antennas (for longer ones this model gives non-trivial angular dependence as one can see from the formula above).

ii) The half-wave dipole antenna: a center-fed antenna of length $L = \lambda/2$. The current distribution is modeled by $I(z) = I_0 \cos kz$. A calculation similar to the one in part i) (it is also done explicitly in Section 8.2.5 of the Steane's book [2]) gives

$$E(r) = \frac{I_0 \cos\left(\frac{\pi}{2}\cos\theta\right)}{2\pi\epsilon_0 c \sin\theta} \frac{e^{i(kr - \omega t)}}{r} . \tag{20}$$

The power is independent of L and λ, and proportional to I_0^2.

Problem 7

Show that the space-space part of the energy-momentum tensor

$$T^{\mu\nu} = \epsilon_0 c^2 \left(-F^{\mu\lambda} F^{\nu}_{\lambda} - \frac{1}{4} g^{\mu\nu} \, F_{\kappa\lambda} F^{\kappa\lambda} \right)$$

is

$$\sigma^{ij} = \frac{1}{2}\epsilon_0 \left(E^k E_k + c^2 B^k B_k \right) \delta^{ij} - \epsilon_0 \left(E^i E^j + c^2 B^i B^j \right)$$

(Greek indices run over space and time, and Latin indices over space only.) Use the stress-energy tensor $T^{\mu\nu}$ to find the forces exerted by the magnetic field inside a long cylindrical solenoid of radius 3 cm and field 1 Tesla. Mu-metal is an alloy of high magnetic permeability that can be used to provide shielding against magnetic fields. If a piece of mu-metal is placed against the end of a solenoid, it "confines" the magnetic field to the interior of the solenoid. By interpreting the stress-energy tensor for the field on each side of the mu-metal sheet, discover whether the latter is attracted or repelled by the solenoid, and find the net force.

Solution:

We use the notation $\eta_{\mu\nu}$ rather than $g_{\mu\nu}$ for the Minkowski metric tensor.

The energy-momentum tensor of electromagnetic field is given by

$$T^{\mu\nu} = \epsilon_0 c^2 \left(-F^{\mu\lambda} \eta_{\lambda\rho} F^{\rho\nu} - \frac{1}{2} \eta^{\mu\nu} D \right) ,$$

where $D = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = \mathbf{B}^2 - \mathbf{E}^2/c^2$ is one of the electromagnetic field invariants. Explicitly,

$$D = \frac{1}{2} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} F_{\mu\nu} F^{\nu\mu} = -\frac{1}{2} Tr(F^{\mu\nu} F_{\nu\kappa}) = -\frac{1}{2} Tr(F^{\mu\nu} \eta_{\nu\rho} F^{\rho\sigma} \eta_{\sigma\kappa}) \, .$$

In matrix form,

$$D = -\frac{1}{2} Tr(F\eta F\eta) = \mathbf{B}^2 - \mathbf{E}^2/c^2 \, .$$

In matrix form, the energy-momentum tensor is written as

$$T = \epsilon_0 c^2 \left(-F\eta F - \frac{1}{2}\eta D \right) = \epsilon_0 c^2 \left(-F\eta F + \frac{1}{4} \eta Tr(F\eta F\eta) \right) \, .$$

Here $\eta = diag(-1, 1, 1, 1)$ and

$$F = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \, .$$

The space-space part (in components: $i, j = x, y, z$) is

$$T^{ij} = \epsilon_0 c^2 \left(-F^{i\mu} \eta_{\mu\nu} F^{\nu j} - \frac{1}{2} \delta^{ij} D \right) = \epsilon_0 c^2 \left(-F^{0i} F^{0j} + F^{ik} F^{jk} - \frac{1}{2} \delta^{ij} D \right) .$$

Since $F^{0i} = E^i/c$ and $F^{ik} = F_{ik} = \epsilon_{ikm} B_m$, we have:

$$\begin{aligned} T^{ij} &= -\epsilon_0 E_i E_j + \epsilon_0 c^2 \left(\delta^{ij} \mathbf{B}^2 - B_i B_j \right) - \frac{\epsilon_0 c^2}{2} \delta^{ij} \left(\mathbf{B}^2 - \mathbf{E}^2/c^2 \right) \\ &= \frac{\epsilon_0 c^2}{2} \delta^{ij} \left(\mathbf{B}^2 - \mathbf{E}^2/c^2 \right) - \epsilon_0 \left(E_i E_j + B_i B_j c^2 \right) = \sigma_{ij} = \sigma^{ij} . \end{aligned}$$

Recall that T_{ij} is the force per unit area in the i-th direction, acting on the surface element oriented in the j-th direction (with orientation fixed by the orientation of a unit normal vector).

For a solenoid oriented on the z direction we have $\mathbf{B} = (0, 0, B)$ and $\mathbf{E} = 0$. We find by a direct substitution

$$T^{\mu\nu} = \frac{\epsilon_0\, c^2 B^2}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

This means that there is an outward force acting in positive x and y directions but an inward force in the z direction. This means that a piece of the mu-metal is attracted at the solenoid's ends.

Note: For electromagnetic field, the trace of the energy-momentum tensor vanishes:

$$T^{\mu}_{\mu} = \eta_{\mu\rho} T^{\rho\mu} = tr(\eta\, T) = 0 \, ,$$

as can be seen from the definition of $T^{\mu\nu}$*.*

Also, one can show that

$$T^{\mu}_{\rho} T^{\rho}_{\nu} = \left(\frac{\epsilon_0 c^2}{2} \right)^2 \delta^{\mu}_{\nu} \cdot K \, ,$$

where $K = \left(\mathbf{B}^2 - \frac{\mathbf{E}^2}{c^2} \right)^2 + (2\, \mathbf{E} \cdot \mathbf{B})^2$.

Problem 8

Write down the stress-energy tensor and the 4-wave vector for an electromagnetic plane wave propagating in the x direction.

Such a wave is observed in two frames in standard configuration. Show that the values of radiation pressure P, momentum density g, energy density u, and frequency ν in the two frames satisfy

$$\frac{P'}{P} = \frac{g'}{g} = \frac{u'}{u} = \frac{\nu'^2}{\nu^2} \,.$$

(Optional: can you prove this for any relative motion of the frame? [Hint: write $T^{\mu\nu}$ in terms of k^μ.]).

A confused student proposes that these quantities should transform like ν'/ν rather than ν'^2/ν^2, on the grounds that energy-momentum $N^\mu = (uc, \mathbf{N})$ is a 4-vector and so should transform in the same way as the wave vector. What is wrong with this argument?

Solution:

For electromagnetic wave propagating in the x direction, we have $k^\mu = (\frac{\omega}{c}, k_x, 0, 0)$, with $-\omega^2/c^2 + k_x^2 = 0$, where $\lambda\nu = c$, $\omega = 2\pi\nu$ and $k_x \equiv k$. The field components are

$$\begin{aligned} E_x &= 0\,, \\ E_y &= E_0 \cos{(kx - \omega t)}\,, \\ E_z &= 0 \end{aligned} \tag{21}$$

and

$$\begin{aligned} B_x &= 0\,, \\ B_y &= 0\,, \\ B_z &= \frac{E_0}{c}\cos{(kx - \omega t)}\,. \end{aligned} \tag{22}$$

Direct substitution gives the energy-momentum tensor

$$T^{\mu\nu} = \epsilon_0\, c^2 E_0^2 \cos^2(kx - \omega t) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{23}$$

In another frame,

$$T'^{\mu\nu} = \Lambda^\mu_{\ \rho}\, T^{\rho\sigma}\, \Lambda_\sigma^{\ \nu}\,.$$

In matrix form, this is

$$T' = \Lambda\, T \Lambda^T \,,$$

where, in the standard case when S' is moving along the x-direction of S,

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

We find

$$T'^{\mu\nu} = \epsilon_0\, c^2 E_0^2 \cos^2(kx - \omega t)\gamma^2(1-\beta)^2 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \tag{24}$$

i.e.

$$T'^{\mu\nu} = \gamma^2(1-\beta)^2 T^{\mu\nu} \,.$$

For the specific scenario we are considering (the wave is propagating along x direction, and S' is moving along x direction), the 3 non-vanishing independent components of $T^{\mu\nu}$ (the energy density $u = T^{00}$, the momentum density component $g_x \equiv g = T^{10} = T^{01}$ and the radiation pressure $P_x \equiv P = T^{xx}$ transform as

$$\frac{u'}{u} = \frac{g'}{g} = \frac{P'}{P} = \gamma^2(1-\beta)^2 \,.$$

At the same time, the 4-vector k^μ transforms as

$$k'^0 = \gamma \left(k^0 - \beta k^1\right) \,,$$

leading to (since $k^2 = 0$, we have $|\mathbf{k}| = \omega/c$):

$$\omega' = \gamma \left(\omega - \beta\omega\right) \,,$$

or (since $\omega = 2\pi\nu$):

$$\nu' = \gamma\,(1-\beta)\,\nu \,.$$

This shows that indeed

$$\frac{\nu'^2}{\nu^2} = \frac{u'}{u} = \frac{g'}{g} = \frac{P'}{P} = \gamma^2(1-\beta)^2 \,. \tag{25}$$

In the case of arbitrary direction of motion of S' relative to S, one may proceed as follows. Eq. (23) can be re-written as

$$T^{\mu\nu} = \epsilon_0\, c^2 E_0^2 \cos^2(kx-\omega t) \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \epsilon_0\, c^4\, \mathbf{E}^2\, \frac{k^\mu k^\nu}{\omega^2}\,. \tag{26}$$

Note that the tensor product of two 4-vectors k^μ and k^ν is a symmetric tensor of rank 2, and there are no other tensorial object of the same type we can built with ingredients charactersising the wave (one may think about $\eta^{\mu\nu}k^2$ but this is zero since $k^2=0$). So we have a correct tensor structure and thus the prefactor must be a scalar (we can call it C), i.e. $T^{\mu\nu} = Ck^\mu k^\nu$. Then e.g. $T^{00} = Ck^0k^0$ and $T'^{00}/T^{00} = \omega'^2/\omega^2$. Also, $C \sim |\mathbf{E}'|^2/\omega'^2 = |\mathbf{E}|^2/\omega^2$. Thus, we find

$$\frac{u'}{u} = \frac{\omega'^2}{\omega^2} = \frac{\nu'^2}{\nu^2} = \frac{|\mathbf{E}'|^2}{|\mathbf{E}|^2}\,. \tag{27}$$

This ratio will in general depend on the angle between the velocity of S' and the wave-vector $\mathbf{k}$. For example, observing the wave in S' (with the source, i.e. S, moving with velocity $\mathbf{v}' = -\mathbf{v}$ relative to S'), and remembering Doppler shift formulas from Problem Set 2, we have

$$\frac{\omega'}{\omega} = \frac{\sqrt{1-\beta^2}}{1+\beta\cos\theta'}\,, \tag{28}$$

where θ' is the angle (in S') between the direction of motion of S in S' (i.e.$\mathbf{v}'$) and the direction of propagation of the wave in S' (i.e. $\mathbf{k}'$). Then

$$\frac{u'}{u} = \frac{\omega'^2}{\omega^2} = \frac{|\mathbf{E}'|^2}{|\mathbf{E}|^2} = \frac{1-\beta^2}{(1+\beta\cos\theta')^2}\,. \tag{29}$$

Similar formulas can be obtained for the components of the momentum density $g_i = T^{0i}$ and pressure $P_x = T^{xx}$, $P_y = T^{yy}$, $P_z = T^{zz}$, again, by comparing the relevant components of $T^{\mu\nu} = Ck^\mu k^\nu$ and $T'^{\mu\nu} = Ck'^\mu k'^\nu$.

Finally, note that the energy-momentum tensor components are *densities* of energy and momentum of the field (energy and momentum are obtained by integrating them). They form a second-rank tensor and transform as components of such a tensor. These components should not be confused with components of a 4-vector. One can also show explicitly that the object N^μ is not a 4-vector (it does not transform as one).

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[1] J. Jackson, *Classical Electrodynamics* (Wiley, 1998).

[2] A. Steane, *Relativity Made Relatively Easy* (Oxford University Press, 2012).