OXFORD UNIVERSITY
PHYSICS DEPARTMENT
3RD YEAR UNDERGRADUATE COURSE

SYMMETRY AND RELATIVITY

TUTORIAL VI

Forces and fields
Problem Set 4
(Part A: problems 1-4)

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Reminder:

Maxwell's equations in $3d$ notations

$$
\begin{aligned}
&\nabla \cdot \mathbf{B} = 0\,, \\
&\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \\
&\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,, \\
&\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}
\end{aligned}
$$

can be written in covariant form as follows (in Minkowski signature $(-+++)$):

$$
\begin{aligned}
&\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu\,, \\
&\partial_\mu \tilde{F}^{\mu\nu} = 0\,,
\end{aligned}
$$

where $J^\nu = (\rho c, \rho \mathbf{v})$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, with $A^\mu = (\phi/c, \mathbf{A})$, i.e.

$$
F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}.
$$

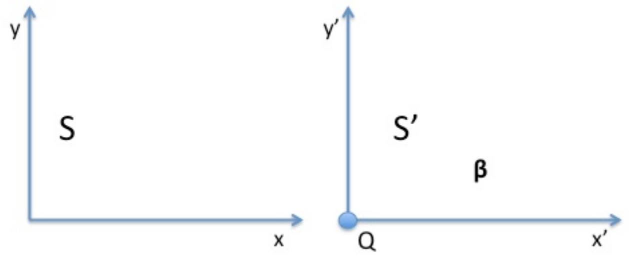
Also, $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda} F^{\kappa\lambda}$, i.e.

$$
\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0\,. \end{pmatrix}.
$$

Problem 1

Obtain the electric field of a uniformly moving charge, as follows: place the charge at the origin of the primed frame S' and write down the field in that frame, then transform to S using the equations for the transformation of the fields (not the force transformation method) and the coordinates. Be sure to write your result in terms of coordinates in the appropriate frame. Sketch the field lines. Prove (from the transformation equations, or otherwise) that the magnetic field of a uniformly moving charge is related to its electric field by $\mathbf{B} = \mathbf{v} \wedge \mathbf{E}/c^2$.

Solution:



In S', the electric field due to a single charge Q' is:

$$\vec{E}' = \frac{Q'}{4\pi\epsilon_0 r'^2} \frac{\vec{r}\,'}{r'} \,. \tag{1}$$

In components, this is $\vec{E}' = (f(r')x', f(r')y', f(r')z')$, where $f(r') = \frac{Q'}{4\pi\epsilon_0 r'^2}$.

Recall the field transformations

$$\mathbf{E}'_{||} = \mathbf{E}_{||} \,, \tag{2}$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||} \,, \tag{3}$$

$$\mathbf{E}'_{\perp} = \gamma \left(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B} \right) \,, \tag{4}$$

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2 \right) \,. \tag{5}$$

Transforming to S, we have: $\vec{E}_{||} = \vec{E}'_{||} \Rightarrow E_x = E'_x$. The charge is invariant, $Q' = Q$, but we need to transform r':

$$\begin{cases} x' = \gamma \left(x - \beta ct \right) = \gamma \left(x - vt \right) \\ ct' = \gamma \left(ct - \beta x \right) \\ y' = y \\ z' = z. \end{cases} \tag{6}$$

Therefore,

$$E_x(t,x,y,z) = \frac{Q\gamma(x-vt)}{4\pi\epsilon_0\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{3/2}}\,. \tag{7}$$

Now,

$$\vec{E}_\perp = \gamma\left(\vec{E}'_\perp - \vec{v}\times\vec{B}'\right) = \gamma\vec{E}'_\perp\,, \tag{8}$$

and hence

$$\begin{aligned} E_y(t,x,y,z) = \gamma E'_y &= \frac{Q\gamma y}{4\pi\epsilon_0\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{3/2}}\,,\\ E_z(t,x,y,z) = \gamma E'_z &= \frac{Q\gamma z}{4\pi\epsilon_0\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{3/2}}. \end{aligned} \tag{9}$$

In particular, at $t=0$:

$$\vec{E} = \frac{Q\gamma\vec{r}}{4\pi\epsilon_0\left(\gamma^2x^2+y^2+z^2\right)^{3/2}}. \tag{10}$$

We now look at the magnetic field:

$$\begin{aligned} &\vec{B}_\parallel = \vec{B}'_| \ \Rightarrow B_x = B'_x = 0,\\ &\vec{B}_\perp = \gamma\left(\vec{B}'_\perp + \vec{v}\times\vec{E}'/c^2\right) = \frac{\gamma}{c^2}\vec{v}\times\vec{E}' = \frac{\gamma}{c^2}\vec{v}\times\vec{E}'_\perp. \end{aligned} \tag{11}$$

Since

$$\begin{aligned} \vec{E}'_\perp &= \frac{1}{\gamma}\vec{E}_\perp \Rightarrow \vec{B}_\perp = \frac{1}{c^2}\vec{v}\times\vec{E}_\perp,\\ \vec{B}_\parallel &= 0 = \frac{1}{c^2}\vec{v}\times\vec{E}_\parallel \equiv 0, \end{aligned} \tag{12}$$

it follows that

$$\vec{B} = \vec{v}\times\vec{E}/c^2. \tag{13}$$

Explicitly, we have

$$\vec{B} = \frac{1}{c^2}\begin{vmatrix} i & j & k\\ v & 0 & 0\\ E_x & E_y & E_z \end{vmatrix} = \left(0, -vE_z/c^2, vE_y/c^2\right)\,, \tag{14}$$

and therefore (taking also into account that $\epsilon_0\mu_0 = 1/c^2$):

$$B_x = 0\,, \tag{15}$$

$$B_y = -\frac{\mu_0 Q\gamma v z}{4\pi\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{3/2}}\,, \tag{16}$$

$$B_z = \frac{\mu_0 Q\gamma v y}{4\pi\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{3/2}}. \tag{17}$$

Problem 2

A sphere of radius a in its rest frame is uniformly charged with charge density $\rho = 3q/4\pi a^3$ *where q is the total charge. Find the fields due to a moving charged sphere by two methods, as follows. [N.B. it will be useful to let the rest frame of the sphere be S' (not S) and to let the frame in which we want the fields be S. This will help to avoid a proliferation of primes in the equations you will be writing down. Let S and S' be in the standard configuration.]*

(i) Field method: write down the electric field as a function of position in the rest frame of the sphere, for the two regions $r' < a$ *and* $r' \geq a$*, where* $r' = (x'^2 + y'^2 + z'^2)^{1/2}$*. Use the field transformation equations to find the electric and magnetic fields in frame S (re-using results from previous questions where possible), making clear in what regions of space your formulae apply.*

(ii) Potential method: in the rest frame of the sphere the 3-vector potential is zero, and the scalar potential is

$$\phi' = \frac{q}{8\pi\epsilon_0 a}\left(3 - r'^2/a^2\right)$$

for $r' < a$*, and*

$$\phi' = \frac{q}{4\pi\epsilon_0 r'}$$

for $r' \geq a$*. Form the 4-vector potential, transform it, and thus show that both* ϕ *and* $\mathbf{A}$ *are time-dependent in frame S. Hence derive the fields for a moving sphere. [Beware when taking gradients that you do not muddle* $\partial/\partial x$, $\partial/\partial x'$*, etc.]*

Solution:

Suppose that the sphere is at rest in S'. The electric field can be found by using the Gauss-Ostrogradsky theorem. For $r' > a$ we have

$$\mathbf{E}' = \frac{q}{4\pi\epsilon_0 r'^2}\frac{\mathbf{r}'}{r'},$$

where $q = q'$ is the total charge and $r' = |\mathbf{r}'|$. For $r' < a$ we get

$$E' 4\pi r'^2 = \frac{Q}{\epsilon_0} = \frac{\rho}{\epsilon_0}\frac{4}{3}\pi r'^3 .$$

Therefore, the electric field in S' is

$$r' > a : \qquad \mathbf{E}' = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}'}{r'^3}, \tag{18}$$

$$r' < a : \qquad \mathbf{E}' = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}'}{a^3}. \tag{19}$$

In S', the sphere is defined by the equation $x'^2 + y'^2 + z'^2 = a^2$, whereas in S, since $x' = \gamma(x - \beta ct)$, $y' = y$, $z' = z$, this becomes $\gamma^2(x - vt)^2 + y^2 + z^2 = a^2$, so in S it looks like a pancake (this is of practical importance for relativistic heavy ion collisions currently ongoing at RHIC and LHC).

Fields *outside* the region in S defined by the equation $\gamma^2(x - vt)^2 + y^2 + z^2 = a^2$ are given by (as found in the previous problem):

$$E_x(t, x, y, z) = \frac{q\gamma(x - vt)}{4\pi\epsilon_0 \left(\gamma^2(x - vt)^2 + y^2 + z^2\right)^{3/2}}\,, \tag{20}$$

$$E_y(t, x, y, z) = \frac{q\gamma y}{4\pi\epsilon_0 \left(\gamma^2(x - vt)^2 + y^2 + z^2\right)^{3/2}}\,, \tag{21}$$

$$E_z(t, x, y, z) = \frac{q\gamma z}{4\pi\epsilon_0 \left(\gamma^2(x - vt)^2 + y^2 + z^2\right)^{3/2}}. \tag{22}$$

$$B_x(t, x, y, z) = 0\,, \tag{23}$$

$$B_y(t, x, y, z) = -\frac{\mu_0 q\gamma v z}{4\pi \left(\gamma^2(x - vt)^2 + y^2 + z^2\right)^{3/2}}\,, \tag{24}$$

$$B_z(t, x, y, z) = \frac{\mu_0 q\gamma v y}{4\pi \left(\gamma^2(x - vt)^2 + y^2 + z^2\right)^{3/2}}. \tag{25}$$

The fields inside the region $\gamma^2(x - vt)^2 + y^2 + z^2 = a^2$ can be found by transforming the electric field $\mathbf{E}' = (kx', ky', kz')$, where $k \equiv q/4\pi\epsilon_0 a^3$, according to

$$\mathbf{E}_{||} = \mathbf{E}'_{||}\,, \tag{26}$$

$$\mathbf{B}_{||} = \mathbf{B}'_{||}\,, \tag{27}$$

$$\mathbf{E}_{\perp} = \gamma\left(\mathbf{E}'_{\perp} - \mathbf{v} \wedge \mathbf{B}'\right)\,, \tag{28}$$

$$\mathbf{B}_{\perp} = \gamma\left(\mathbf{B}'_{\perp} + \mathbf{v} \wedge \mathbf{E}'/c^2\right)\,. \tag{29}$$

We obtain for the electric field

$$E_x = E'_x = kx' = k\gamma(x - vt)\,, \tag{30}$$

$$E_y = \gamma E'_y = k\gamma y' = k\gamma y\,, \tag{31}$$

$$E_z = \gamma E'_z = k\gamma z' = k\gamma z\,, \tag{32}$$

and, since

$$\mathbf{v} \times \mathbf{E}' = \begin{vmatrix} i & j & k \\ v & 0 & 0 \\ E'_x & E'_y & E'_z \end{vmatrix} = \left(0, -vE'_z, vE'_y\right)\,, \tag{33}$$

for the magnetic field we have

$$B_x = B'_x = 0\,, \tag{34}$$

$$B_y = -\gamma v E'_z/c^2 = -k\gamma v z/c^2\,, \tag{35}$$

$$B_z = \gamma v E'_y/c^2 = k\gamma v y/c^2\,. \tag{36}$$

One can see that $\mathbf{B} = \mathbf{v}\times\mathbf{E}/c^2$:

$$\mathbf{B} = \mathbf{v}\times\mathbf{E}/c^2 = \frac{1}{c^2}\begin{vmatrix} i & j & k \\ v & 0 & 0 \\ E_x & E_y & E_z \end{vmatrix} = \left(0, -vE_z/c^2, vE_y/c^2\right) = \left(0, -\gamma v k z/c^2, \gamma v k y/c^2\right), \tag{37}$$

as found earlier.

(ii) In frame S', the 4-potential is given by $A'^{\mu} = (\phi'/c, \mathbf{A}')$, where $\mathbf{A}' = 0$ and

$$r' < a: \qquad \phi' = \frac{q}{8\pi\epsilon_0 a}\left(3 - r'^2/a^2\right)\,, \tag{38}$$

$$r' \geq a: \qquad \phi' = \frac{q}{4\pi\epsilon_0 r'}\,. \tag{39}$$

In the region outside the sphere we have therefore

$$A'^{\mu} = \left(\frac{q}{4\pi\epsilon_0 r' c}, 0, 0, 0\right)\,.$$

Applying Lorentz transformations to the 4-vector A'^{μ}, we find

$$A^0 = \gamma\left(A'^0 + \beta A'^1\right) = \frac{\gamma q}{4\pi\epsilon_0 r' c}$$

and

$$A^1 = \gamma\left(A'^1 + \beta A'^0\right) = \frac{\gamma\beta q}{4\pi\epsilon_0 r' c}\,.$$

Therefore,

$$A^{\mu} = \left(\frac{\gamma q}{4\pi\epsilon_0 r' c}, \frac{\gamma\beta q}{4\pi\epsilon_0 r' c}, 0, 0\right)\,,$$

where $r'^2 = \gamma^2(x-vt)^2 + y^2 + z^2$. We now find components of electric field. Since $\mathbf{E} = -\nabla\phi - \partial_t\mathbf{A}$, we have $E_x = -\partial_x\phi - \partial_t A_x$. Now,

$$\phi = \frac{\gamma q}{4\pi\epsilon_0 c}\left(\gamma^2(x-vt)^2 + y^2 + z^2\right)^{-1/2}$$

and

$$\partial_x\phi = -\frac{\gamma q}{4\pi\epsilon_0 r' c}\gamma^2(x-vt)\left(\gamma^2(x-vt)^2 + y^2 + z^2\right)^{-3/2}\,,$$

$$\partial_t A_x = \frac{\gamma\beta q}{4\pi\epsilon_0 c}\gamma^2 v(x-vt)\left(\gamma^2(x-vt)^2+y^2+z^2\right)^{-3/2} \, ,$$

so

$$E_x = \frac{\gamma q(x-vt)}{4\pi\epsilon_0\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} \, .$$

Similarly, $E_y = -\partial_y\phi - \partial_t A_y$,

$$E_y = \frac{\gamma q y}{4\pi\epsilon_0\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}}$$

and $E_z = -\partial_z\phi - \partial_t A_z$,

$$E_z = \frac{\gamma q z}{4\pi\epsilon_0\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} \, .$$

The magnetic field is given by $\mathbf{B} = \nabla \times \mathbf{A}$, i.e.

$$\mathbf{B} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix} = \left(0,\ \frac{\gamma\beta q}{4\pi\epsilon_0 c}\partial_z\frac{1}{r'},\ -\frac{\gamma\beta q}{4\pi\epsilon_0 c}\partial_z\frac{1}{r'}\right) \, , \tag{40}$$

thus,

$$B_x(t,x,y,z) = 0 \, , \tag{41}$$

$$B_y(t,x,y,z) = -\frac{q\gamma v z}{4\pi\epsilon_0 c^2\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} \, , \tag{42}$$

$$B_z(t,x,y,z) = \frac{q\gamma v y}{4\pi\epsilon_0 c^2\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} . \tag{43}$$

Using $\epsilon_0\mu_0 = 1/c^2$, the expressions for the magnetic field components can be written as

$$B_x(t,x,y,z) = 0 \, , \tag{44}$$

$$B_y(t,x,y,z) = -\frac{\mu_0 q\gamma v z}{4\pi\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} \, , \tag{45}$$

$$B_z(t,x,y,z) = \frac{\mu_0 q\gamma v y}{4\pi\left[\gamma^2(x-vt)^2+y^2+z^2\right]^{3/2}} . \tag{46}$$

In the region *inside* the sphere,

$$A'^\mu = \left(\frac{q}{8\pi\epsilon_0 ac}\left(3 - \frac{r'^2}{a^2}\right), 0, 0, 0\right) \, ,$$

where $r'^2 = \gamma^2(x-vt)^2 + y^2 + z^2$. Making the Lorentz transformation, we find

$$A^0 = \gamma A'^0 = \frac{\gamma q}{8\pi\epsilon_0 ac}\left(3 - \frac{r'^2}{a^2}\right) \, , \tag{47}$$

$$A^1 = \gamma\beta A'^0 = \frac{\gamma\beta q}{8\pi\epsilon_0 ac}\left(3 - \frac{r'^2}{a^2}\right) \, . \tag{48}$$

With $E_x = -\partial_x \phi - \partial_t A_x$, one finds

$$-\partial_x \phi = \partial_x \left(\frac{q\gamma r'^2}{8\pi\epsilon_0 a^3} \right) = \frac{\gamma q}{4\pi\epsilon_0 a^3} \gamma^2 (x - vt) \,,$$

$$-\partial_t A_x = \frac{\gamma q \beta}{8\pi\epsilon_0 a^3 c} \partial_t r'^2 = -\frac{q\gamma^3 (x - vt)\beta^2}{4\pi\epsilon_0 a^3} \,,$$

and therefore

$$E_x = \frac{q\gamma(x - vt)}{4\pi\epsilon_0 a^3} = k\gamma(x - vt) \,.$$

Similarly, $E_y = -\partial_y \phi - \partial_t A_y$, $E_z = -\partial_z \phi - \partial_t A_z$, so

$$E_y = -\partial_y \phi = \frac{q\gamma}{8\pi\epsilon_0 a^3} \partial_y r'^2 = \frac{q\gamma y}{4\pi\epsilon_0 a^3} \,,$$

$$E_z = -\partial_z \phi = \frac{q\gamma z}{4\pi\epsilon_0 a^3} \,,$$

i.e.

$$\mathbf{E} = (k\gamma(x - vt),\, k\gamma y,\, k\gamma z) \,,$$

which coincides with Eqs. (30)—(32).

The magnetic field is computed as follows:

$$\mathbf{B} = \nabla \times \mathbf{A} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ A_x & 0 & 0 \end{vmatrix} \,, \tag{49}$$

leading to

$$B_x = 0 \,, \tag{50}$$

$$B_y = \partial_z A_x = -\frac{q\gamma\beta}{8\pi\epsilon_0 a^3 c} \partial_z r'^2 = -\frac{q\gamma v z}{4\pi\epsilon_0 a^3 c^2} \,, \tag{51}$$

$$B_z = -\partial_y A_x = \frac{q\gamma\beta}{8\pi\epsilon_0 a^3 c} \partial_y r'^2 = \frac{q\gamma v y}{4\pi\epsilon_0 a^3 c^2} \,, \tag{52}$$

i.e.

$$\mathbf{B} = (0, -\gamma v k z / c^2, \gamma v k y / c^2) \,,$$

which coincides with Eqs. (34)—(36).

Problem 3

In a frame S a point charge first moves uniformly along the negative x-axis in the positive x-direction, reaching the point $(-d,0,0)$ at $t=-\Delta t$, and then it is slowed down until it comes to rest at the origin at $t=0$. Sketch the lines of electric field in S at $t=0$, in the region $(x+d)^2+y^2+z^2>(c\Delta t)^2$.

Solution: The motion of the charge in S is characterised by $x(t)=-d+v_0(t+\Delta t)$ for $t<-\Delta t$. We don't know exactly how the charge was brought to rest (and this will not be essential) but for the sake of illustration we can assume that it was done by applying some constant force. Then $x(t)=-dt^2/(\Delta t)^2$ for $-\Delta t<t<0$, and $x(t)=0$ for $t>0$. Velocity is given by

$$v_x(t)=\begin{cases} v_0\,, \text{ for } t<-\Delta t\,, \\ -\frac{2dt}{(\Delta t)^2}\,, \text{ for } -\Delta t<t<0\,, \\ 0, \text{ for } t>0\,, \end{cases}$$

where $v_0=2d/\Delta t$. Had the charge continued to move with constant velocity after $t=-\Delta t$, at $t=0$ it would be at $x_*=-d+v_0\Delta t>0$ (in our illustration, $x_*=d$). Clearly, x_* is inside the sphere $(x+d)^2+y^2+z^2>(c\Delta t)^2$, since for $x=x_*$ (and $y=0$, $z=0$) we have $(x+d)^2=(v_0\Delta t)^2<(c\Delta t)^2$ as $v_0<c$.

Outside the light front sphere $(x+d)^2=(v_0\Delta t)^2=(c\Delta t)^2$ fields "don't know yet" that the charge started to decelerate, so they are fields of a charge moving uniformly according to the law $x(t)=-d+v_0(t+\Delta t)$.

In S' (associated with the charge), the potential is

$$\phi'(\mathbf{r}',t')=\frac{q}{4\pi\epsilon_0 r'}\,.$$

Since

$$A^0=\gamma\left(A'^0+\beta A'^1\right)=\gamma A'^0\,, \tag{53}$$

$$A^1=\gamma\left(A'^1+\beta A'^0\right)=\gamma\beta A'^0\,, \tag{54}$$

and also $x'=\gamma(x+d-v_0\Delta t-v_0 t)$, $y'=y$, $z'=z$ (note the need to shift the coordinate in the Lorentz transformation to have $x'=0$ at $t=-\Delta t$, $x=-d$), we get

$$\phi(\mathbf{r},t)=\frac{\gamma q}{4\pi\epsilon_0\sqrt{\gamma^2(x-v_0t+d-v_0\Delta t)^2+y^2+z^2}}\,, \tag{55}$$

$$\mathbf{A}(\mathbf{r},t)=\left(\frac{v_0}{c^2}\phi(\mathbf{r},t),\ 0,\ 0\right)\,. \tag{56}$$

The components of electric field are computed as

$$E_x = -\partial_t A_x - \partial_x \phi = -\frac{v_0}{c^2}\partial_t \phi - \partial_x \phi\,, \tag{57}$$

$$E_y = -\partial_y \phi \tag{58}$$

$$E_z = -\partial_z \phi\,. \tag{59}$$

Taking the derivatives, we find

$$E_x = -\frac{\gamma q}{4\pi\epsilon_0}\frac{x - v_0 t + d - v_0\Delta t}{[\gamma^2(x - v_0 t + d - v_0\Delta t)^2 + y^2 + z^2]^{3/2}}\,, \tag{60}$$

$$E_y = -\frac{\gamma q}{4\pi\epsilon_0}\frac{y}{[\gamma^2(x - v_0 t + d - v_0\Delta t)^2 + y^2 + z^2]^{3/2}}\,, \tag{61}$$

$$E_z = -\frac{\gamma q}{4\pi\epsilon_0}\frac{z}{[\gamma^2(x - v_0 t + d - v_0\Delta t)^2 + y^2 + z^2]^{3/2}}\,. \tag{62}$$

Consider for simplicity the $x - y$ plane only . Note that

$$\frac{E_y(t=0)}{E_x(t=0)} = \frac{y}{x - x_0}\,,$$

where $x_0 = -d + v_0\Delta t > 0$. So for the components of $\mathbf{E}$, $\tan\varphi = y/(x - x_0)$. Thus, the lines of electric field *outside* the lightfront sphere are those of a point charge q located at $x = x_0$. The field is stronger in the direction closer to the y-axis, as the formulas suggest.

Problem 4

Give a 4-vector argument to show that the 4-vector potential of a point charge q in an arbitrary state of motion is given by

$$A^\mu = \frac{q}{4\pi\epsilon_0}\frac{U^\mu/c}{(-R_\nu U^\nu)}\,,$$

where U^μ and R^μ are suitably chosen 4-vectors which you should define in your answer.

Solution:

A charge q moving in an inertial reference frame S according to $\mathbf{r}_0 = \mathbf{r_0}(t)$ is a source of electromagnetic field determined at the point of observation $\mathbf{r}$ by a 4-vector potential

$$A^\mu = \frac{q}{4\pi\epsilon_0}\frac{U^\mu/c}{(-R_\nu U^\nu)}\,, \tag{63}$$

where $U^\mu = dx^\mu/d\tau = (\gamma c, \gamma \mathbf{v}_0)$, with $\mathbf{v}_0 = d\mathbf{r_0}/dt$. The 4-vector R^μ has components $\mathbf{R} = \mathbf{r} - \mathbf{r}_0$, $R^0 = cT$, where $T = R/c$, and $R = |\mathbf{R}|$. First, we need to understand the ingredients. We have

$$-R_\nu U^\nu = cT\gamma c - \gamma \mathbf{R}\mathbf{v}_0\,,$$

so, in components,

$$\phi(\mathbf{r},t) = cA^0 = \frac{q}{4\pi\epsilon_0}\frac{1}{R - \mathbf{R}\mathbf{v}_0/c}\,, \tag{64}$$

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu_0 q}{4\pi}\frac{\mathbf{v}_0}{R - \mathbf{R}\mathbf{v}_0/c}\,. \tag{65}$$

In these formulas, $\mathbf{r}_0$ and $\mathbf{v}_0$ are functions of τ_R determined by the equation

$$\tau_R = t - \frac{|\mathbf{r} - \mathbf{r}_0(\tau_R)|}{c}$$

as a function of t and $\mathbf{r}$. One may note also that here

$$\mathbf{A} = \frac{\mathbf{v}_0}{c}A^0 = \frac{\mathbf{v}_0}{c}\frac{\phi}{c}\,.$$

These expressions are known as Lienard-Wiechert potentials. In the rest frame of the particle, $\mathbf{v_0} = 0$, $\mathbf{r_0} = 0$, and we have

$$\phi(\mathbf{r},t) = cA^0 = \frac{q}{4\pi\epsilon_0|\mathbf{r}|}\,, \qquad \mathbf{A} = 0\,. \tag{66}$$

One can derive Eqs. (64)—(65) by solving Maxwell's equations. Alternatively, one can write down the covariant form (63) by noticing that the covariant objects describing the physics in this situation are the 4-vectors R^μ and U^μ, and that their combination (necessarily unique, since A^μ is a tensor) giving the correct expression in particle's rest frame (66) is given by (63). One has to be careful here, since in the case of arbitrary motion, other 4-vectors (e.g. acceleration A^μ) exist, and one might want to include terms involving e.g. derivatives of U^μ: if such terms give zero in the rest frame, there is *a priori* no reason not to include them.