OXFORD UNIVERSITY
PHYSICS DEPARTMENT
3RD YEAR UNDERGRADUATE COURSE

SYMMETRY AND RELATIVITY

TUTORIAL V

Forces and fields

Problem Set 3

(Part B: problems 5-9)

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Problem 5

Show that two of Maxwell's equations are guaranteed to be satisfied if the fields are expressed in terms of potentials $\mathbf{A}$ *and* ϕ *such that*

$$\mathbf{B} = \nabla \wedge \mathbf{A}\,, \tag{1}$$

$$\mathbf{E} = -\left(\frac{\partial \mathbf{A}}{\partial t}\right) - \nabla\phi\,. \tag{2}$$

(i) Express the other two of Maxwell's equations in terms of $\mathbf{A}$ *and* ϕ*.*

(ii) Introduce a gauge condition to simplify the equations, and hence express Maxwell's equations in terms of 4-vectors, 4-vector operators, and Lorentz scalars (a manifestly covariant form).

Solution:

We may start by recalling the Maxwell's equations in $3d$ notations:

$$\nabla \cdot \mathbf{B} = 0\,, \tag{3}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}\,, \tag{4}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\,, \tag{5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\,. \tag{6}$$

There are 4 equations altogether (8 in terms of components), appearing in two groups (each containing two equations), with a prominent asymmetry between the groups. The first group (Eqs. (3), (4)) contains no sources (ρ and $\mathbf{J}$). These equations must be satisfied automatically, regardless of the distribution of charges and currents in space-time. Such equations are known as *constraints*. They often imply the presence of a symmetry in the theory under consideration.

If $\mathbf{E}$ and $\mathbf{B}$ are written in terms of $\mathbf{A}$ and ϕ as in Eqs. (1), (2), then the equations (3), (4) are automatically satisfied. Indeed, since $B_i = \varepsilon_{ijk}\partial_j A_k$, we have $\nabla \cdot \mathbf{B} = \partial_i B_i = \varepsilon_{ijk}\partial_i\partial_j A_k = 0$, since the contraction of ij in ε_{ijk} and ij in a symmetric object ($\partial_i\partial_j = \partial_j\partial_i$) gives zero.

Note that all manipulations are done in $3d$ Euclidean space, where there is no need to distinguish between lower and upper indices.

Explicitly, $\varepsilon_{ijk}\partial_i\partial_j A_k = \varepsilon_{jik}\partial_j\partial_i A_k$ [we re-labeled $i \to j$ and $j \to i$] $= \varepsilon_{jik}\partial_i\partial_j A_k$ [since $\partial_i\partial_j = \partial_j\partial_i$] $= -\varepsilon_{ijk}\partial_i\partial_j A_k$. We arrived at $\varepsilon_{ijk}\partial_i\partial_j A_k = -\varepsilon_{ijk}\partial_i\partial_j A_k$, which implies $\varepsilon_{ijk}\partial_i\partial_j A_k = 0$, since $X = -X$ implies $X = 0$.

Now consider $(\nabla \times \mathbf{E})_i = \varepsilon_{ikl}\partial_k E_l = -\varepsilon_{ikl}\partial_k(\partial_t A_l + \partial_l\phi) = -\partial_t\varepsilon_{ikl}\partial_k A_l - \varepsilon_{ikl}\partial_k\partial_l\phi = -\partial_t B_i$, since $\varepsilon_{ikl}\partial_k\partial_l\phi \equiv 0$ for reasons discussed above. Thus, equations (3), (4) are automatically satisfied.

(i) We now write the other two Maxwell's equations in terms of $\mathbf{A}$ and ϕ.

Since $\nabla \cdot \mathbf{E} = -\partial_t \partial_i A_i - \partial^2 \phi$, where $\partial^2 = \partial_i \partial_i = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian in $3d$, Eq. (5) becomes

$$- \partial_t \partial_i A_i - \partial^2 \phi = \frac{\rho}{\epsilon_0} \,. \tag{7}$$

Computing the curl, we find $(\nabla \times \mathbf{B})_i = \varepsilon_{ijk} \partial_j E_k = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \partial_l A_m = \varepsilon_{ijk} \varepsilon_{klm} \partial_j \partial_l A_m = \partial_i \partial_m A_m - \partial^2 A_i$, where we used the identity $\varepsilon_{kij} \varepsilon_{klm} = \delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}$. Therefore, Eq. (6) in components becomes

$$\partial_i \partial_m A_m - \partial^2 A_i = \mu_0 J_i - \mu_0 \epsilon_0 (\partial_t^2 A_i + \partial_i \partial_t \phi) \tag{8}$$

and can be re-written as (taking into account that $\epsilon_0 \mu_0 = 1/c^2$)

$$- \frac{1}{c^2} \partial_t^2 A_i + \partial^2 A_i - \frac{1}{c^2} \partial_i \partial_t \phi - \partial_i \partial_m A_m = -\mu_0 J_i \,. \tag{9}$$

(ii) Introducing the 4-vector $A^\mu = (\phi/c, \mathbf{A})$, we can write Eqs. (7), (9) in the form

$$-\partial^2 \phi - \partial_t \partial_i A_i = \frac{\rho}{\epsilon_0} \,, \tag{10}$$

$$-\partial_i (\partial_\mu A^\mu) + \Box A_i = -\mu_0 J_i \,, \tag{11}$$

where

$$\Box \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu = -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \partial^2 \tag{12}$$

is the d'Alembertian in $4d$ Minkowski space and

$$\partial_\mu A^\mu = \frac{\partial A^0}{\partial x^0} + \frac{\partial A^i}{\partial x^i} = \frac{1}{c^2} \frac{\partial \phi}{\partial t} + \mathrm{div} \mathbf{A} \,. \tag{13}$$

The important fact is that the correspondence between A^μ and the fields $\mathbf{E}$ and $\mathbf{B}$ is not one to one: there are (infinitely) many potentials A^μ corresponding to the same values of $\mathbf{E}$ and $\mathbf{B}$. All such equivalent A^μ are related by

$$A^\mu \to A^\mu - \partial^\mu f(x) \,, \tag{14}$$

where f is a smooth function (this, of course, can be shown explicitly, by using definitions of $\mathbf{E}$ and $\mathbf{B}$ via ϕ and A^i). One can say that the whole *orbit* of A^μ parametrised by f corresponds to the same values of $\mathbf{E}$ and $\mathbf{B}$. This phenomenon is known as *gauge invariance* and the transformation (14) as *gauge transformation.* Electromagnetism is the simplest example of a gauge theory.

We can choose a representative on the orbit of A^μ by *fixing a gauge* (or choosing a gauge condition). One popular gauge condition is $\partial_\mu A^\mu = 0$, known as the Lorentz gauge. In the Lorentz gauge, Maxwell's equations (10), (11) have a very simple form

$$\Box\phi = -\frac{\rho}{\epsilon_0}\,, \tag{15}$$

$$\Box A_i = -\mu_0 J_i\,. \tag{16}$$

Of course, these equations should be supplemented by the appropriate boundary and/or initial conditions. Note that the equations are *linear* PDEs.

Problem 6

How does a second rank tensor changes under a Lorentz transformation? By transforming the field tensor and interpreting the result, prove that the electromagnetic field transforms as

$$\mathbf{E}'_{||} = \mathbf{E}_{||}\,, \tag{17}$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||}\,, \tag{18}$$

$$\mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v}\wedge\mathbf{B}\right)\,, \tag{19}$$

$$\mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - \mathbf{v}\wedge\mathbf{E}/c^2\right)\,. \tag{20}$$

[Hint: you may find the algebra easier if you treat **E** *and* **B** *separately. Do you need to work out all the matrix elements, or can you argue that you already know the symmetry?]*

Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.

Solution:

The field strength tensor is given by

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix} \tag{21}$$

(Note that the signs of matrix elements are sensitive to the choice of the Minkowski metric convention - here we use $(-+++)$, but for $(+---)$ all entries change sign.)

Since $F^{\mu\nu}$ is a tensor of rank $(2,0)$, under general continuous coordinate transformation $x \to x' = x'(x)$ it transforms as

$$F'^{\mu\nu} = \frac{\partial x'^{\mu}}{\partial x^{\rho}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}F^{\rho\sigma}\,. \tag{22}$$

Lorentz transformations are linear, $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}$. For the motion of S' along OX we have $x'^0 = \gamma(x^0 - \beta x^1)$, $x'^1 = \gamma(x^1 - \beta x^0)$, i.e.

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{23}$$

In matrix form, the transformation (22) is

$$F' = \Lambda F \Lambda^T\,. \tag{24}$$

(This may require a bit of attention - note that (22) is written for individual components of the matrices; convince yourself, maybe using simple $2d$ examples, that (24) is correct.)

The result of the matrix multiplication is $F' = A + B$, where

$$A = \begin{pmatrix} 0 & E_x/c & \gamma E_y/c & \gamma E_z/c \\ -E_x/c & 0 & -\beta\gamma E_y/c & -\beta\gamma E_z/c \\ -\gamma E_y/c & \beta\gamma E_y/c & 0 & 0 \\ -\gamma E_z/c & \beta\gamma E_y/c & 0 & 0 \end{pmatrix} \tag{25}$$

and

$$B = \begin{pmatrix} 0 & 0 & -\beta\gamma B_z & \beta\gamma B_y \\ 0 & 0 & \gamma B_z & -\gamma B_y \\ \beta\gamma B_z & -\gamma B_z & 0 & B_x \\ -\beta\gamma B_y & \gamma B_y & -B_x & 0 \end{pmatrix} . \tag{26}$$

Comparing this with the standard form of $F^{\mu\nu}$, we see that $E'_x = E_x$, $E'_y = \gamma E_y - \gamma\beta c B_z$, $E'_z = \gamma E_z + \gamma\beta c B_y$. Since the motion is along OX, $E_x = E_{||}$, and $E_y, E_z = E_\perp$. Thus, we can write the above formulas as

$$\mathbf{E}'_{||} = \mathbf{E}_{||} \,, \tag{27}$$

$$\mathbf{E}'_\perp = \gamma\left(\mathbf{E}_\perp + \mathbf{v} \wedge \mathbf{B}\right) . \tag{28}$$

Similarly, $B'_x = B_x$, $B'_y = \gamma B_y + \gamma\beta E_z/c$, $B'_z = \gamma B_z - \gamma\beta E_y/c$, which can be written in the form

$$\mathbf{B}'_{||} = \mathbf{B}_{||} \,, \tag{29}$$

$$\mathbf{B}'_\perp = \gamma\left(\mathbf{B}_\perp - \mathbf{v} \wedge \mathbf{E}/c^2\right) . \tag{30}$$

We now apply these formulas to a concrete problem: *Find the magnetic field due to a long straight current by Lorentz transformation from the electric field due to a line charge.*

We use the field transformations (written here in slightly different notations)

$$\begin{aligned} \vec{E}'_{||} &= \vec{E}_{||} \,, \\ \vec{E}'_\perp &= \gamma\left(\vec{E}_\perp + \vec{v} \times \vec{B}\right) \,, \\ \vec{B}'_{||} &= \vec{B}_{||} \,, \\ \vec{B}'_\perp &= \gamma\left(\vec{B}_\perp - \vec{v} \times \vec{E}/c^2\right) . \end{aligned} \tag{31}$$

In the frame S where the charge carriers are not moving, the magnetic field is zero and the electric

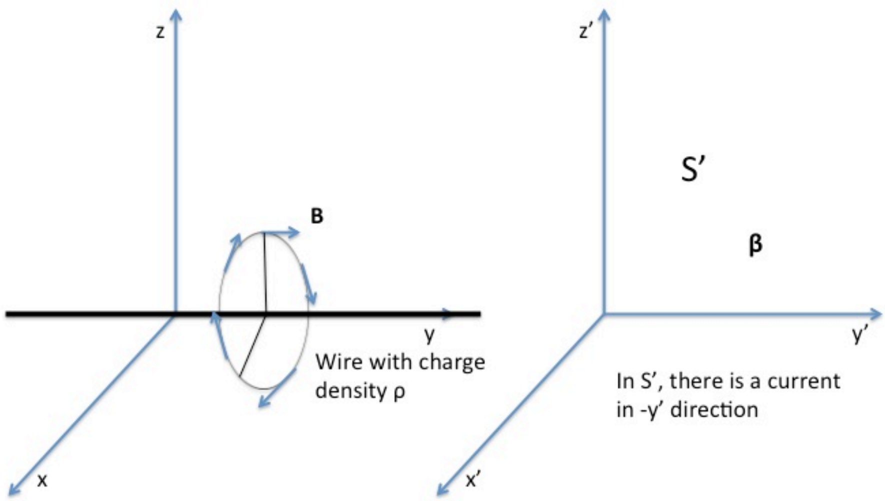

FIG. 1: Problem 6: the line charge viewed in different reference frames.

field $\vec{E}$ can be found by using Gauss-Ostrogradsky's theorem[1]:

$$\vec{E} = \frac{\rho}{2\pi\epsilon_0 r}\vec{n}, \tag{32}$$

where $r = \sqrt{x^2 + y^2}$, $|\vec{n}| = 1$, $\vec{E} = (E_x, 0, E_z)$.

We can now transform to S' moving along OY with velocity $\vec{v}$ (thus in S' there is a current flowing in the negative y' direction). In S':

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} = 0, \qquad \vec{E}'_{\perp} = \gamma\vec{E}_{\perp}, \tag{33}$$

$$\vec{B}'_{\parallel} = 0, \qquad \vec{B}'_{\perp} = -\gamma\vec{v}\times\vec{E}/c^2, \tag{34}$$

Therefore, the magnitude of the magnetic field in S' is

$$|\vec{B}'_{\perp}| = \frac{\gamma}{c^2}v\frac{\rho}{2\pi\epsilon_0 r}. \tag{35}$$

In Eq. (35), we need to express all quantities via the corresponding quantities in S'. Note that $r' = r$ (x, z do not transform), and $\rho' dl' = \rho dl$ (charge conservation), therefore $\rho' = \gamma\rho$ (since $dl' = dl/\gamma$). Also, $\vec{v}' = -\vec{v}$. Thus

$$|\vec{B}'_{\perp}| = \frac{v'\rho'}{2\pi\epsilon_0 r' c^2} = \frac{I'}{2\pi\epsilon_0 r' c^2}, \tag{36}$$

where $I' = \rho' v'$. Note that $c^2 = 1/\epsilon_0\mu_0$, hence

$$|\vec{B}'_{\perp}| = \frac{\mu_0 I'}{2\pi r'}, \tag{37}$$

as anticipated.

[1] First established by Lagrange. This is a special case of the general Stokes' formula for differential forms.

Problem 7

The electromagnetic field tensor $F^{\mu\nu}$ (sometimes called the Faraday tensor) is defined such that the 4-force on a charged particle is given by

$$f^{\mu} = qF^{\mu\nu}U_{\nu}\,. \tag{38}$$

By comparing this to the Lorentz force equation

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \wedge \mathbf{B}\right)$$

which defines the electric and magnetic fields (keeping in mind the distinction between $d\mathbf{p}/dt$ and $dP^{\mu}/d\tau$), show that the components of the field tensor are

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_x/c & B_y & -B_x & 0 \end{pmatrix}.$$

Solution:

We can recall that the 4-velocity $U^{\mu} = (\gamma c, \gamma \mathbf{v})$ and the components of the 4-force are

$$\frac{dp^{\mu}}{d\tau} = f^{\mu} = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma \mathbf{f}\right),$$

where $p^{\mu} = (E/c, \mathbf{p})$, $d\tau = dt/\gamma$ and $dp_i/dt = f_i$ (or $dp^i/dt = f^i$; note that $f_i = f^i$).

The $3d$ equation of motion involving Lorentz force is

$$\dot{p}^i = qE^i + q(v \times B)^i\,. \tag{39}$$

On the other hand, Eq. (38) gives

$$\gamma \dot{p}^i = -q\gamma c F^{i0} + q\gamma F^{ik} v_k\,. \tag{40}$$

Comparing Eqs. (39) and (40), we find $F^{i0} = -E^i/c = -E_i/c$.

Since $U_{\mu}f^{\mu} = 0$ for $m = const$ (this can be seen by remembering that $U_{\mu}f^{\mu}$ is Lorentz-invariant and computing it in particle's rest frame, where $U_{\mu}f^{\mu} = -c^2\dot{m} = 0$ for $m = const$), we have $F^{\mu\nu}U_{\mu}U_{\nu} = 0$ and hence $F^{\mu\nu} = -F^{\nu\mu}$, since $U_{\mu}U_{\nu} = U_{\nu}U_{\mu}$. Thus, $F^{\mu\nu}$ is antisymmetric, and $F^{0i} = -F^{i0} = E^i/c = E_i/c$.

Comparing now the part with v^i, we find $\varepsilon_{ijk}v_jB_k = F_{ij}v_j$ (note that with $i, j, k = x, y, z$ we have $F_{ij} = F^{ij}$). Therefore, $F^{ij} = F_{ij} = \varepsilon_{ijk}B_k$, i.e. $F^{12} = \varepsilon_{123}B_3 = B_z$, $F^{13} = \varepsilon_{132}B_2 = -B_y$ and $F^{23} = \varepsilon_{231}B_1 = B_x$.

Thus, with all diagonal components vanishing due to antisymmetry, we get

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{41}$$

Problem 8

Show that the field equation $\partial_\lambda F^{\lambda\nu} = -\mu_0 \rho_0 U^\nu$ *is equivalent to*

$$\partial^\lambda \partial_\lambda A^\nu - \partial^\nu \left(\partial_\lambda A^\lambda\right) = -\mu_0 J^\nu \,,$$

where $J^\nu \equiv \rho_0 U^\nu$ *(here* ρ_0 *is the proper charge density, and* J^ν *is the 4-current density). Comment.*

Solution:

Since $F^{\lambda\nu} = \partial^\lambda A^\nu - \partial^\nu A^\lambda$, we get

$$\partial_\lambda \partial^\lambda A^\nu - \partial^\nu \left(\partial_\lambda A^\lambda\right) = -\mu_0 J^\nu \,, \tag{42}$$

where $J^\nu = \rho_0 U^\nu$, with $U^\nu = (\gamma c, \gamma \mathbf{v})$, so that $J^\nu = (\gamma\rho_0 c, \gamma\rho_0 \mathbf{v})$. Note that ρ_0 is the proper charge density, i.e. the charge density in the rest frame of the charge. We can introduce $\rho = \gamma\rho_0$, then $J^\nu = (\rho c, \rho\mathbf{v})$.

In the Lorentz gauge ($\partial_\mu A^\mu = 0$) Eq. (43) becomes

$$\partial_\lambda \partial^\lambda A^\nu \equiv \Box A^\nu = -\mu_0 J^\nu \,, \tag{43}$$

In components:

$$\Box A^0 = -\mu_0 J^0 = -\mu_0 \rho c \,, \tag{44}$$

or, since $A^\mu = (\phi/c, \mathbf{A})$,

$$\Box\phi = -\frac{\rho}{\epsilon_0} \,. \tag{45}$$

The remaining components in Eq. (43) are

$$\Box A^i = -\mu_0 J^i \,. \tag{46}$$

Eqs. (45) and (46) are Maxwell's equations in the Lorentz gauge.

One can also show (do this) that the other pair of Maxwell's equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$, i.e. the equations without sources, can be written as $\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0$, or, equivalently, as $\partial_\mu \tilde{F}^{\mu\nu} = 0$.

The Moral: Maxwell's equations in covariant form are written as

$$\partial_\mu F^{\mu\nu} = -\mu_0 J^\nu \,, \tag{47}$$

$$\partial_\mu \tilde{F}^{\mu\nu} = 0 \,. \tag{48}$$

Note that the sign in (47) corresponds to signature $(-+++)$.

Problem 9

Show that the following two scalar quantities are Lorentz invariant: $D = \mathbf{B}^2 - \mathbf{E}^2/c^2$ *and* $\alpha = \mathbf{B}\cdot\mathbf{E}/c$. *[Hint: for the second, introduce the dual field tensor* $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$.*]*

Show that if $\alpha = 0$ *but* $D \neq 0$ *then either there is a frame in which the field is purely electric, or there is a frame in which the field is purely magnetic. Give the condition required for each case, and find an example of such a frame (by specifying its velocity relative to one in which the fields are* $\mathbf{E}$, $\mathbf{B}$*). Suggest a type of field for which both* $\alpha = 0$ *and* $D = 0$.

Solution:

First, we show that $D = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}$ and $\alpha = \frac{1}{4}\tilde{F}_{\mu\nu}F^{\mu\nu}$. We have

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{49}$$

Lowering the indices with Minkowski tensor, we find $F_{\mu\nu} = \eta_{\mu\lambda}\,\eta_{\nu\sigma}\,F^{\lambda\sigma}$. Explicitly,

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & B_z & -B_y \\ E_y/c & -B_z & 0 & B_x \\ E_z/c & B_y & -B_x & 0 \end{pmatrix}. \tag{50}$$

Note that $F_{\mu\nu}F^{\mu\nu} = -F_{\mu\nu}F^{\nu\mu} = -trF_{\mu\nu}F^{\nu\sigma}$. So, to compute $F_{\mu\nu}F^{\mu\nu}$, one has to multiply the matrices (49) and (50) and compute the trace (with the minus sign) of the resulting matrix. This gives

$$F_{\mu\nu}F^{\mu\nu} = 2\mathbf{B}^2 - 2\mathbf{E}^2/c^2 = 2D\,. \tag{51}$$

To get α, one uses the definition $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\kappa\lambda}F^{\kappa\lambda}$. Note that $\tilde{F}_{\mu\nu} = -\tilde{F}_{\nu\mu}$. Explicitly, one has $\tilde{F}_{01} = \frac{1}{2}\epsilon_{01\mu\nu}F^{\mu\nu} = \frac{1}{2}\epsilon_{0123}F^{23} + \frac{1}{2}\epsilon_{0132}F^{32} = \frac{1}{2}\epsilon_{0123}F^{23} - \frac{1}{2}\epsilon_{0123}F^{32} = \frac{1}{2}(F^{23} - F^{32}) = F^{23} = B_x$, and similarly for other components. The result is

$$\tilde{F}_{\mu\nu} = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & E_z/c & -E_y/c \\ -B_y & -E_z/c & 0 & E_x/c \\ -B_z & E_y/c & -E_x/c & 0\,. \end{pmatrix}. \tag{52}$$

Again, computing $\tilde{F}_{\mu\nu}F^{\mu\nu} = -\tilde{F}_{\mu\nu}F^{\nu\mu} = -tr\tilde{F}_{\mu\nu}F^{\nu\sigma} = \frac{4}{c}\vec{E}\cdot\vec{B} = 4\alpha$.

$$\tilde{F}_{\mu\nu}F^{\mu\nu} = \frac{4}{c}\,\vec{E}\cdot\vec{B} = 4\,\alpha\,. \tag{53}$$

One can check that $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ gives the same as $F_{\mu\nu}F^{\mu\nu}$, i.e. it is not a new invariant. Note also that $\tilde{F}_{\mu\nu}$ is a pseudotensor (and, correspondingly, α is a pseudoscalar), i.e. they change sign under parity transformation P: $\mathbf{x} \to -\mathbf{x}$.

If $\alpha = 0$, then $\vec{E} \cdot \vec{B} = 0$, i.e. in a given frame S fields $\vec{E}$ and $\vec{B}$ are either perpendicular, or one of them is zero. In the latter case, the problem is solved in S. In the former case, the outcome depends on the sign of $D \neq 0$.

1) $D < 0$: This means that in S we have $\mathbf{B}^2 < \mathbf{E}^2/c^2$. Then one can find a frame S' with $\mathbf{B}' = 0$. Recall the field transformations

$$\mathbf{E}'_{||} = \mathbf{E}_{||} \,, \tag{54}$$

$$\mathbf{B}'_{||} = \mathbf{B}_{||} \,, \tag{55}$$

$$\mathbf{E}'_{\perp} = \gamma \left(\mathbf{E}_{\perp} + \mathbf{v} \wedge \mathbf{B} \right) \,, \tag{56}$$

$$\mathbf{B}'_{\perp} = \gamma \left(\mathbf{B}_{\perp} - \mathbf{v} \wedge \mathbf{E}/c^2 \right) \,. \tag{57}$$

Choose S' moving with $\mathbf{v} \perp \mathbf{B}$. Then $\mathbf{B}_{||} = 0$ (and thus $\mathbf{B}'_{||} = 0$) and $\mathbf{B}_{\perp} = \mathbf{B}$. Choose the magnitude of $\mathbf{v}$ so that $\mathbf{v} \times \mathbf{E}/c^2 = \mathbf{B}$, i.e. $v = Bc^2/E$, where E and B are the magnitudes of the fields. Then is S' we have $\mathbf{B}'_{||} = 0$ and $\mathbf{B}'_{\perp} = 0$, so there is only an electric field in S'. Note that, since $D < 0$, we have $\beta = v/c = Bc/E < 1$, so such a frame exists.

2) $D > 0$: This means that in S we have $\mathbf{B}^2 > \mathbf{E}^2/c^2$. We can find a frame S' where the field is purely magnetic. Choosing $\mathbf{v} \perp \mathbf{E}$, so that $\mathbf{E}_{||} = 0$ (and thus $\mathbf{E}'_{||} = 0$), and the magnitude of the velocity so that $\mathbf{v} \times \mathbf{B} = -\mathbf{E}_{\perp}$, i.e. $v = E/B$, we find that in S' only the magnetic field is non-zero. Such a frame exists, since $v/c = E/Bc < 1$, since $D > 0$.

3) The important case of $D = 0$ and $\alpha = 0$ corresponds to the solution of Maxwell's equations known as electromagnetic waves.