OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

# SYMMETRY AND RELATIVITY

TUTORIAL IV

**Forces and fields**

**Problem Set 3**

**(Part A: problems 1-4)**

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**Problem 1**

*Obtain the transformation equations for 3-force, by starting from the Lorentz transformation of energy-momentum, and then differentiating with respect to $t'$. [Hint: argue that the relative velocity $\vec{V}$ of the reference frame is constant, and use or derive an expression for $dt/dt'$.]*

**Solution:**

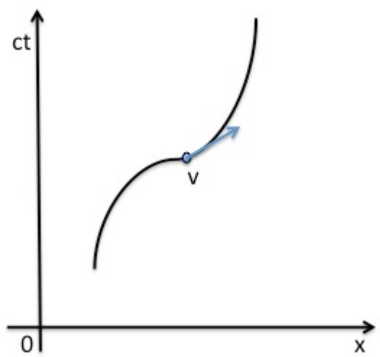
In non-relativistic physics, the second Newton's law is $d\vec{p}/dt = \vec{f}$. In special relativity, this is replaced by

$$\frac{dp^\mu}{d\tau} = F^\mu \,, \tag{1}$$

where $p^\mu = (E/c, \vec{p})$ is a 4-vector and $d\tau = dt/\gamma$ is the proper time of the particle. By taking the derivative on the left-hand-side of eq. (1) explicitly, we find

$$F^\mu = \left(\frac{\gamma}{c}\dot{E}, \gamma\frac{d\vec{p}}{dt}\right) = \left(\frac{\gamma}{c}\dot{E}, \gamma\vec{f}\right) \,, \tag{2}$$

where $\dot{E} \equiv dE/dt$ and $\gamma = \gamma(v)$, where $\vec{v}$ is the velocity of the particle in the original inertial reference frame $S$.




We now consider a frame $S'$ moving with constant velocity $\vec{V}$ with respect to $S$. We are interested in the components $\vec{f}'$ of the 3-force in that frame. In principle, we can find $\vec{f}'$ by transforming the 4-vector (2) to $S'$. However, it is more convenient to work with the left-hand-side of eq. (1). Indeed, considering for simplicity the motion of $S'$ along $x$ in $S$, we have

$$p'_x = \gamma(V)\left(p_x - \beta_V\frac{E}{c}\right) \,, \tag{3}$$

$$p'_y = p_y \,, \tag{4}$$

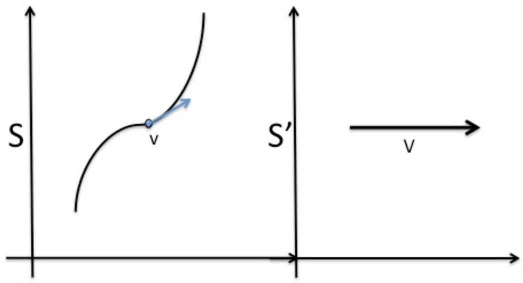
$$p'_z = p_z \,, \tag{5}$$

where $\beta_V = |\vec{V}|/c$ and $\gamma(V) = 1/\sqrt{1-\beta_V^2}$ are the constant Lorentz factors associated with the motion of $S'$. We have then

$$f'_x = \frac{dp'_x}{dt'} = \gamma(V)\left(\frac{dp_x}{dt}\frac{dt}{dt'} - \beta_V \frac{\dot{E}}{c}\frac{dt}{dt'}\right) = \gamma(V)\left(f_x - \beta_V \frac{\dot{E}}{c}\right)\frac{dt}{dt'} , \tag{6}$$

$$f'_y = \frac{dp'_y}{dt'} = \frac{dp_y}{dt}\frac{dt}{dt'} = f_y \frac{dt}{dt'} , \tag{7}$$

$$f'_z = \frac{dp'_z}{dt'} = \frac{dp_z}{dt}\frac{dt}{dt'} = f_z \frac{dt}{dt'} . \tag{8}$$




To find $dt'/dt$, we use the Lorentz transformation for $t'$:

$$ct' = \gamma(V)\left(ct - \beta_V x\right) , \tag{9}$$

which gives

$$\frac{dt'}{dt} = \gamma(V)\left(1 - \beta_V v_x\right) = \gamma(V)\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right) , \tag{10}$$

where $v_x = dx/dt$. Finally, we obtain

$$f'_x = \frac{f_x - \beta_V \frac{\dot{E}}{c}}{\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)} = \frac{f_x - \beta_V \frac{\vec{f}\cdot\vec{v}}{c}}{\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)} , \tag{11}$$

$$f'_y = \frac{f_y}{\gamma(V)\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)} , \tag{12}$$

$$f'_z = \frac{f_z}{\gamma(V)\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)} . \tag{13}$$

Note that this transformation is similar to the transformation of the particle's velocity components

$$v'_x = \frac{v_x - V}{\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)}, \tag{14}$$

$$v'_y = \frac{v_y}{\gamma(V)\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)}, \tag{15}$$

$$v'_z = \frac{v_z}{\gamma(V)\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)}. \tag{16}$$

If $S'$ is moving in an arbitrary direction w.r.t. $S$ with constant velocity $\vec{V}$, the force can be decomposed into the parts parallel and perpendicular to the direction of $\vec{V}$, $\vec{f} = \vec{f}_{||} + \vec{f}_{\perp}$, eqs. (6) — (8) generalize to

$$\vec{f}'_{||} = \frac{\vec{f}_{||} - \vec{V}\frac{\vec{f}\cdot\vec{v}}{c^2}}{\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)}, \tag{17}$$

$$\vec{f}'_{\perp} = \frac{\vec{f}_{\perp}}{\gamma(V)\left(1 - \frac{\vec{V}\cdot\vec{v}}{c^2}\right)}. \tag{18}$$

**Problem 2**

*Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows:*

*(i) Write down the solution for* $\mathbf{p}$ *as a function of time, taking as initial condition* $\mathbf{p}(0) = \mathbf{p_0}$*.*

*(ii) Show that the Lorentz factor as a function of time is given by* $\gamma^2 = 1 + \alpha^2$*, where* $\boldsymbol{\alpha} = (\mathbf{p}_0 + \mathbf{f}t)/mc$*.*

*(iii) You can now write down the solution for* $\mathbf{v}$ *as a function of time. Do so.*

*(iv) Now restrict attention to the case where* $\mathbf{p_0}$ *is perpendicular to* $\mathbf{f}$*. Taking the x-direction along* $\mathbf{f}$ *and the y-direction along* $\mathbf{p_0}$*, show that the trajectory is given by*

$$\begin{aligned} x &= \frac{c}{f}\left(m^2c^2 + p_0^2 + f^2t^2\right)^{1/2} + const\,, \\ y &= \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + const\,, \end{aligned} \tag{19}$$

*where you may quote that* $\int (a^2+t^2)^{-1/2}dt = \log\left[t + \sqrt{a^2+t^2}\right]$*.*

*(v) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation. [N.B. The calculation as a function of proper time is best done another way, see later problems.]*

**Solution:**

(i) The equation of motion $\dot{\vec{p}} = \vec{f} = const$ is easily integrated to give $\vec{p} = \vec{f}\,t + \vec{p}_0$.

(ii) For a particle of mass $m$, we have $E = \gamma mc^2$ and $\vec{p} = \gamma m\vec{v}$. This implies

$$\frac{\vec{p}^{\,2}}{c^2} = \gamma^2 m^2 \frac{\vec{v}^{\,2}}{c^2} = \gamma^2 m^2 \beta^2\,. \tag{20}$$

Since $1 - \beta^2 = 1/\gamma^2$, we have $\gamma^2 = 1 + \frac{\vec{p}^{\,2}}{m^2c^2}$, i.e.

$$\gamma^2 = 1 + \frac{(\vec{f}\,t + \vec{p}_0)^2}{m^2c^2}\,. \tag{21}$$

(iii) Velocity can be found as

$$\vec{v} = \frac{\vec{p}}{\gamma m} = \frac{\vec{f}\,t + \vec{p}_0}{m}\frac{1}{\sqrt{1 + \frac{(\vec{f}\,t+\vec{p}_0)^2}{m^2c^2}}}\,. \tag{22}$$

(iv) With $\vec{f} = (f, 0, 0)$ and $\vec{p}_0 = (0, p_0, 0)$, we have the velocity components as

$$\begin{aligned} \dot{x} &= v_x = \frac{ft}{m\sqrt{1 + \frac{f^2t^2 + p_0^2}{m^2c^2}}}\,, \\ \dot{y} &= v_y = \frac{p_0}{m\sqrt{1 + \frac{f^2t^2 + p_0^2}{m^2c^2}}}\,, \\ \dot{z} &= v_z = 0\,. \end{aligned} \tag{23}$$

Integrating this, we find

$$
\begin{aligned}
x(t) &= \frac{mc^2}{f}\left[\sqrt{1+\frac{f^2t^2+p_0^2}{m^2c^2}}-\sqrt{1+\frac{p_0^2}{m^2c^2}}\right]+x_0\,,\\
y(t) &= \frac{cp_0}{f}\log\left[\frac{ft+\sqrt{m^2c^2+p_0^2+f^2t^2}}{\sqrt{m^2c^2+p_0^2}}\right]+y_0\,,\\
z(t) &= z_0\,. \qquad (24)
\end{aligned}
$$

(v) In the general case (when $\vec{p}_0$ is not orthogonal to $\vec{f}$), we can choose the coordinate system S in such a way that vectors $\vec{p}_0$ and $\vec{f}$ belong to the $x-y$ plane, with the $x$-axis in the direction of $\vec{f}$. Then we can Lorentz-transform to $S'$ moving with velocity $V = v_{0x} = p_{0x}/\gamma_0 m$ along the $x$-direction of S. Note that the transformation of the force components (11)—(13) then ensures that in S' $\vec{f}' = (f, 0, 0)$. We also have $\vec{p}_0' = (0, p_{0y}, 0)$, where $p_{0y}$ is the $y$-component of the initial momentum in S. This reduces the problem to the previous one.

**Comments:**

- Problem 2 illustrates a rather typical situation in relativistic dynamics: given external forces, you are supposed to find a trajectory of a particle. Of particular importance is the motion of charged particles in external fields (electric and magnetic). Problem 2 can be considered as dealing with the motion of a charged particle in a constant uniform electric field. You may think of answering an additional question: what is the trajectory $y = y(x)$ of the particle in such field? More generally, considering the relevant situations (electric, magnetic fields or both) is strongly recommended: see e.g. chapters 20-22 of [1] or a similar source. Note in particular that part (v) of the present problem mimics the situation when both electric and magnetic fields are present, since, as we shall learn later in the course, one can often eliminate one of them by switching to a suitable reference frame S'.

- A note about the integrals involved: in this problem, we encountered the integral of the type

$$I = \int \frac{dx}{\sqrt{ax^2+bx+c}}\,. \qquad (25)$$

By completing the square and changing variables, we can write $I$ as

$$I = \frac{1}{\sqrt{a}}\int \frac{dz}{\sqrt{z^2-\alpha^2}} = \frac{1}{\sqrt{a}}\int \frac{dt}{\sqrt{t^2-1}}\,, \qquad (26)$$

where $z = x + b/2a$, $\alpha^2 = (b^2-4ac)/4a^2$ and $t = z/\alpha$. We now recall that $\cosh^2\xi - \sinh^2\xi = 1$ and change variables: $t = \cosh\xi$. Then

$$I = \frac{1}{\sqrt{a}}\int d\xi = \frac{1}{\sqrt{a}}\operatorname{arcosh} t = \frac{1}{\sqrt{a}}\log\left[t+\sqrt{t^2-1}\right]. \qquad (27)$$

The transition to the last expression, known as the "long logarithm", is easily proven by using definitions of the hyperbolic functions. Of course, it may happen that $\alpha^2 < 0$. Then we have

$$I = \frac{1}{\sqrt{a}} \int \frac{dt}{\sqrt{1-t^2}} \tag{28}$$

and the appropriate change of variables is $t = \cos\xi$. The integral of this type arises when one computes the length of an arc of a unit circle: indeed, with $y(x) = \sqrt{1-x^2}$,

$$l = \int ds = \int \sqrt{dx^2 + dy^2} = \int dx \sqrt{1 + y'^2} = \int \frac{dx}{\sqrt{1-x^2}} = \arccos x \,. \tag{29}$$

An interesting situation arises when one tries to generalise this to computing the arc length of an ellipse. This leads to the integrals of the type

$$J = \int \frac{dx}{\sqrt{(1-a^2x^2)(1+b^2x^2)}} \tag{30}$$

elliptic functions (doubly periodic generalisation of the trigonometric functions), Mordell hypothesis, Fermat Last Theorem and other fascinating things (see e.g. Chapter 7 in the popular book [2]).

**Problem 3**

*For motion under pure (rest mass preserving) inverse square law force* $\mathbf{f} = -\alpha\mathbf{r}/r^3$*, where* $\alpha$ *is a constant, derive the energy equation* $\gamma mc^2 - \alpha/r = constant$*.*

**Solution:**

The expression for the 4-force is

$$F^\mu = \frac{dP^\mu}{d\tau} = \left(\gamma \frac{\dot{E}}{c}, \gamma \vec{f}\right) ,$$

and the 4-velocity is $u^\mu = (\gamma c, \gamma \vec{v})$. One can form an invariant

$$F_\mu u^\mu = \gamma^2 \left(-\dot{E} + \vec{f} \cdot \vec{v}\right) .$$

In particle's rest frame, $F_0^\mu = (\dot{m}c, \vec{f}_0)$ and $u^\mu = (c, 0)$, so $F_0^\mu u_{\mu,0} = -\dot{m}c^2 = 0$ (by assumption, mass is constant - i.e. fuel of a space-ship is not leaking out etc). This implies $\dot{E} = \vec{f} \cdot \vec{v}$. For a potential force, $\vec{f} = -\nabla U$, and, since $v^i \partial U/\partial x^i = dU/dt$, we have

$$\frac{d}{dt} (E + U) = 0$$

or $E + U = const$. Here, $E = \gamma mc^2$, $U = -\alpha/r$, so

$$\gamma mc^2 - \alpha/r = constant .$$

**Problem 4**

*Prove that the time rate of change of the angular momentum* $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$ *of a particle about an origin* $O$ *is equal to the couple* $\mathbf{r} \wedge \mathbf{f}$ *of the applied force about* $O$.

*If* $L^{\mu\nu}$ *is a particle's 4-angular momentum, and we define the 4-couple* $G^{\mu\nu} \equiv X^\mu F^\nu - X^\nu F^\mu$, *prove that* $(d/d\tau)L^{\mu\nu} = G^{\mu\nu}$, *and that the space-space part of this equation corresponds to the previous 3-vector result.*

**Solution:**

Taking the time derivative of $\mathbf{L} = \mathbf{r} \wedge \mathbf{p}$, we find

$$\frac{d}{dt}\vec{L} = \dot{\vec{r}} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{f},$$

since $\dot{\vec{r}}$ and $\vec{p}$ are collinear.

Then, for $L^{\mu\nu} = X^\mu P^\nu - X^\nu P^\mu$, we have

$$\frac{d}{d\tau}L^{\mu\nu} = \frac{dX^\mu}{d\tau}P^\nu + X^\mu \frac{dP^\nu}{d\tau} - \frac{dX^\nu}{d\tau}P^\mu - X^\nu \frac{dP^\mu}{d\tau} = U^\mu P^\nu + X^\mu \frac{dP^\nu}{d\tau} - U^\nu P^\mu - X^\nu \frac{dP^\mu}{d\tau}\,.$$

Since $P^\mu = mU^\mu$, this reduces to

$$\frac{d}{d\tau}L^{\mu\nu} = X^\mu \frac{dP^\nu}{d\tau} - X^\nu \frac{dP^\mu}{d\tau} = X^\mu F^\nu - X^\nu F^\mu \equiv G^{\mu\nu}\,.$$

Also, $X^\mu = (ct, x^i)$, and

$$F^\mu = \left(\frac{1}{c}\frac{dE}{d\tau}, \frac{d\vec{p}}{d\tau}\right) = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\vec{f}\right)\,.$$

Thus,

$$\gamma \frac{dL^{ij}}{dt} = \gamma\left(x^i f^j - x^j f^i\right)\,,$$

which is the desired result, with the identification $L_i = \varepsilon_{ijk}L^{jk}$.

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[1] L. Landau and E. Lifshitz, *The Classical Theory of Fields: Volume 2 (Course of Theoretical Physics)* (Butterworth-Heinemann, 1980).

[2] I. Stewart, *The great mathematical problems* (Profile Books Ltd, 2014).