OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

SYMMETRY AND RELATIVITY

TUTORIAL II

Kinematics and dynamics

Problem Set 2

(Part B: problems 5-9)

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Problem 5

The 4-vector field F^μ is biven by $F^\mu = 2x^\mu + k^\mu (x^\nu x_\nu)$ where k is a constant 4-vector and $x^\mu = (ct, x, y, z)$ is the 4-vector displacement in spacetime.

Evaluate the following:

(i) $\partial_\lambda x^\lambda$

(ii) $\partial^\mu \left(x_\lambda x^\lambda\right)$

(iii) $\partial^\mu \partial_\mu x^\nu x_\nu$

(iv) $\partial_\lambda F^\lambda$

(v) $\partial^\mu \left(\partial_\lambda F^\lambda\right)$

(vi) $\partial^\mu \partial_\mu \sin k_\lambda x^\lambda$

(vii) $\partial^\mu x^\nu$.

Solution:

It is important to clearly understand the meaning of all the notations. We have $x^\mu = (ct, x^i)$ and $\partial_\mu = \frac{\partial}{\partial x^\mu}$. Then $x_\mu \equiv \eta_{\mu\nu} x^\nu$, where $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski space metric tensor. Thus, $x_\mu = (-ct, x^i)$. (Note that often, e.g. in particle physics, a different signature convention is used, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and formulas will change accordingly; physical results, of course, remain the same.)

(i)

$$\partial_\lambda x^\lambda = \frac{\partial x^\lambda}{\partial x^\lambda} = \partial_0 x^0 + \partial_1 x^1 + ... = 4\,.$$

Alternatively, since

$$\partial_\mu x^\nu = \delta^\nu_\mu\,,$$

we have $\partial_\lambda x^\lambda = \delta^\lambda_\lambda = 4$.

(ii)

$$\partial^\mu \left(x_\lambda x^\lambda\right) = \eta^{\mu\nu} \partial_\nu \left(\eta_{\rho\sigma} x^\rho x^\sigma\right) = \eta^{\mu\nu} \eta_{\rho\sigma} \left(\delta^\rho_\nu x^\sigma + x^\rho \delta^\sigma_\nu\right) = \eta^{\mu\nu} \eta_{\nu\sigma} x^\sigma + \eta^{\mu\nu} \eta_{\rho\nu} x^\rho = 2x^\mu\,.$$

(iii) Note that $\partial^\mu (x_\nu x^\nu) = 2x^\mu$ and $\partial_\mu x^\mu = 4$. Combining these results, we find

$$\partial^\mu \partial_\mu x^\nu x_\nu = 8.$$

(iv) Similarly,

$$\partial_\lambda F^\lambda = 2\partial_\lambda x^\lambda + k^\lambda \partial_\lambda (x^\nu x_\nu) = 8 + 2k^\lambda x_\lambda\,.$$

(v)

$$\partial^\mu(\partial_\lambda F^\lambda) = 2k^\lambda \delta^\mu_\lambda = 2k^\mu \,.$$

(vi)

$$\partial^\mu \partial_\mu (\sin k_\lambda x^\lambda) = \partial^\mu \left(k_\mu \cos k_\lambda x^\lambda \right) = -k^\mu k_\mu \sin k_\lambda x^\lambda = -k^2 \sin kx \,.$$

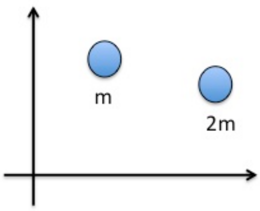
(vii)

$$\partial^\mu x^\nu = \eta^{\mu\rho} \partial_\rho x^\nu = \eta^{\mu\rho} \delta^\nu_\rho = \eta^{\mu\nu} \,.$$

Problem 6

A particle of rest mass m and kinetic energy $3mc^2$ strikes a stationary particle of rest mass $2m$ and combines with it while still conserving energy and momentum. Find the rest mass and speed of the composite particle.

Solution:



In a given inertial frame, the 4-momenta of the two initial particles are $p_1^\mu = (E_1/c, \vec{p}_1)$, $p_2^\mu = (2mc, 0)$. We also know that $E_1 = mc^2 + T = mc^2 + 3mc^2 = 4mc^2$. Since

$$-\frac{E_1^2}{c^2} + |\vec{p}_1|^2 = -m^2c^2\,,$$

we find that $|\vec{p}_1| = \sqrt{15}mc$. Note that $p_1^2 = -m^2c^2$ and $p_2^2 = -4m^2c^2$.

Since the 4-momentum is conserved in the collision, we have

$$p_1 + p_2 = p_f\,.$$

This implies

$$(p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = p_f^2 = -M^2c^2\,,$$

where M is the mass of the composite particle. Explicitly,

$$-m^2c^2 - 4m^2c^2 - 2\frac{E_1}{c}2mc = -M^2c^2\,.$$

Since $E_1 = 4mc^2$, we find $M = \sqrt{21}m$.

For the energy over c, i.e. for the zeroth component of the equation

$$p_1^\mu + p_2^\mu = p_f^\mu$$

we have $E_1/c + 2mc = E_f/c$, so $E_f = 6mc^2$. Since

$$E_f = \gamma_f M c^2$$

we get $\gamma_f = 6/\sqrt{21}$ and therefore $\beta_f = \sqrt{15}/6 \Rightarrow v_f = 0.645c$.

Problem 7

Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction in terms of E_0 and the electron rest mass m.

High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4} eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

Solution:

The 4-momenta of the two photons can be written as $k_1^\mu = \left(\frac{\hbar\omega_1}{c}, \hbar\vec{k_1}\right)$ and $k_2^\mu = \left(\frac{\hbar\omega_2}{c}, \hbar\vec{k_2}\right)$, where $k_1^2 = 0$, $k_2^2 = 0$ (since photons are massless), and $\hbar\omega_1 = E_0$ and $\hbar\omega_2 = E$. Conservation of energy and momentum implies

$$k_1^\mu + k_2^\mu = p_1^\mu + p_2^\mu \,,$$

where p_1 and p_2 are the 4-momenta of the electron and positron. We have then

$$(k_1 + k_2)^2 = 2k_1 \cdot k_2 = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 \,. \tag{1}$$

The invariants can be computed in any inertial frame. It is convenient to compute the invariant on the RHS of eq. (1) in the CMF of the electron-positron pair. The *threshold* condition means that in the CMF, the spatial momenta of electron and positron are zero (the energy is just enough to produce them, but not enough to set them in motion). So, in the CMF: $p_1 = (mc, 0)$ and $p_2 = (mc, 0)$. Eq. (1) becomes

$$2k_1 \cdot k_2 = -2\frac{EE_0}{c^2}(1 - \cos\phi) = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = -4m^2c^2 \,, \tag{2}$$

where ϕ is the angle between the directions of the photons. Thus,

$$E = \frac{2m^2c^4}{E_0(1 - \cos\phi)} \,.$$

Since we are interested in the threshold (minimum) energy $E = E_{min}$, we choose $\phi = \pi$. Then

$$E_{min} = \frac{m^2c^4}{E_0} \,.$$

With the numbers given in the problem, we find $E_{min} \approx 1.14 \times 10^{15}$ eV ~ 1 PeV $\sim 10^3$ TeV, about 100 times more energetic than LHC.

Problem 8

A particle Y decays into three other particles, with labels indicated by $Y \to 1+2+3$. Working throughout in the CM frame:

(i) Show that the 3-momenta of the decay products are coplanar.

(ii) Show that the energy of particle 3 is given by

$$E_3 = \frac{(m_Y^2 + m_3^2 - m_1^2 - m_2^2)c^4 - 2E_1 \cdot E_2 + 2\mathbf{p_1} \cdot \mathbf{p_2}\, c^2}{2m_Y c^2}$$

(iii) Show that the maximum value of E_3 is

$$E_{3,max} = \frac{m_Y^2 + m_3^2 - (m_1 + m_2)^2}{2m_Y}\, c^2 \, .$$

(iv) Show that, when particle 3 has its maximum possible energy, particle 1 has the energy

$$E_1 = \frac{m_1(m_Y c^2 - E_{3,max})}{m_1 + m_2} \, .$$

[Hint: first argue that 1 and 2 have the same speed in this situation.]

(v) Now let's return to the more general circumstance, with E_3 not necessarily maximal. Let X be the system composed of particles 1 and 2. Show that its rest mass is given by

$$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2 \, .$$

(vi) Write down an expression for the energy E^* of particle 2 in the rest frame of X in terms of m_1, m_2 and m_X.

(vii) Show that when particle 3 has an intermediate energy, $m - 3c^2 < E_3 < E_{3,max}$, the energy of particle 2 in the original frame (the rest frame of Y) is in the range

$$\gamma\,(E^* - \beta p^* c) \le E_2 \le \gamma\,(E^* + \beta p^* c) \ ,$$

where E^* and p^* are the energy and momentum of particle 2 in the X-frame and γ and β refer to the speed of that frame relative to the rest frame of Y.

Solution:

(i) The 4-momentum is conserved,

$$p_Y = p_1 + p_2 + p_3 \, , \tag{3}$$

Moreover, in CMF we have $p_Y = (m_Y c, 0)$ and $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$. To show that the three vectors are coplanar, we need to check the condition $\vec{p}_3 \cdot (\vec{p}_1 \times \vec{p}_2) = 0$ (the order of indices 1,2,3 is irrelevant here). Since $\vec{p}_1 = -\vec{p}_2 - \vec{p}_3 = 0$, we have

$$\vec{p}_3 \cdot (\vec{p}_1 \times \vec{p}_2) = -\vec{p}_3 \left[(\vec{p}_2 + \vec{p}_3) \times \vec{p}_2\right] = \vec{p}_3 \cdot (\vec{p}_2 \times \vec{p}_3) \equiv 0.$$

(ii) From eq. (3) we get

$$(p_Y - p_3)^2 = (p_1 + p_2)^2 \; . \tag{4}$$

In CMF, $p_Y = (m_Y c, 0)$, $p_i = (\frac{E_i}{c}, \vec{p}_i)$, $i = 1, 2, 3$, with $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$. Eq. (4) gives

$$-m_Y^2 c^2 - m_3^2 c^2 + 2m_Y E_3 = -m_1^2 c^2 - m_2^2 c^2 - 2\frac{E_1 E_2}{c^2} + 2\mathbf{p_1} \cdot \mathbf{p_2} \, ,$$

from which we find

$$E_3 = \frac{(m_Y^2 + m_3^2 - m_1^2 - m_2^2)c^4 - 2E_1 \cdot E_2 + 2\mathbf{p_1} \cdot \mathbf{p_2} \, c^2}{2m_Y c^2} \; . \tag{5}$$

(iii) Eq. (5) suggests that as far as the angle θ between $\vec{p}_1$ and $\vec{p}_2$ is concerned, the maximal value of E_3 is attained when $\theta = 0$. The other parameters we can maximize with respect to are the magnitudes p_1 and p_2 of the two vectors. The condition $\partial E_3 / \partial p_1 = 0$ implies

$$\frac{\partial E_1}{\partial p_1} E_2 = p_2 c^2$$

or, since $E_1 = \sqrt{p_1^2 c^2 + m_1^2 c^4}$,

$$\frac{p_1}{E_1} = \frac{p_2}{E_2} \; . \tag{6}$$

Also, since $p_1 = \gamma_1 m_1 v_1$ and $E_1 = \gamma_1 m_1 c^2$, we have $p_1 c / E_1 = v_1 / c$. Thus, from eq. (6), we obtain $v_1 = v_2 \equiv v$, i.e. the particles 1 and 2 move with the same speed v in the same direction (opposite to the direction of particle 3). We should also check that the extremum $\partial E_3 / \partial p_1 = 0$ is actually a maximum. This can be done by computing the second derivative (it is rather straightforward to do) and showing that $\partial^2 E_3 / \partial p_1^2 < 0$. Starting with p_2 insetad of p_1 gives the same result. Now, with $E_1 = \gamma(v) m_1 c^2$, $E_2 = \gamma(v) m_2 c^2$, $p_1 = \gamma(v) m_1 v$, $p_2 = \gamma(v) m_2 v$, we get

$$-2\frac{E_1 E_2}{c^2} + 2\mathbf{p_1} \cdot \mathbf{p_2} = -2\gamma^2 m_1 m_2 c^4 + 2\gamma^2 m_1 m_2 \beta^2 c^4 = 2\gamma^2 m_1 m_2 c^4 (1 - \beta^2) = 2m_1 m_2 c^4 \, .$$

Eq. (5) then becomes

$$E_{3,max} = \frac{m_Y^2 + m_3^2 - (m_1 + m_2)^2}{2m_Y} \, c^2 \, .$$

(iv) From conservation of energy (zeroth component of eq. (3)), we can write

$$m_Y c^2 = E_1 + E_2 + E_{3,max} \, ,$$

where all energies are computed in CMF. From the discussion in **(iii)**, we see that $E_2 = m_2 E_1 / m_1$ in this particular situation. Thus,

$$E_1 = \frac{m_1 (m_Y c^2 - E_{3,max})}{m_1 + m_2} \, .$$

(v) Treating 1 and 2 as a composite particle X, instead of eq. (3) we have

$$p_Y = p_X + p_3 \,.$$

Then

$$(p_Y - p_3)^2 = -m_Y^2 c^2 - m_3^2 c^2 + 2E_3 m_Y = p_X^2 = -m_X^2 c^2 \,.$$

Therefore,

$$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2 \,.$$

(vi) Since $p_X = p_1 + p_2$, we can write

$$p_1^2 = (p_X - p_2)^2$$

and compute the right hand side in the rest frame of X, where $p_X = (m_X c, 0)$ and $p_2 = (E^*/c, \vec{p}_*)$. We have

$$-m_1^2 c^2 = -m_X^2 c^2 - m_2^2 c^2 + 2E^* m_X \,,$$

and so

$$E^* = \frac{m_X^2 + m_2^2 - m_1^2}{2m_X} c^2 \,.$$

(vii) In **(vi)**, we found the energy of particle 2, E^*, in the rest frame of X. In the rest frame of Y, the energy E_2 can be found by making a Lorentz transformation from the rest frame of X to the rest frame of Y,

$$E_2 = \gamma \left(E^* + \beta p_{\|}^* c \right) \,,$$

where γ and β refer to the velocity of X w.r.t. Y and $p_{\|}^*$ is the component of $\vec{p}_*$ parallel to that velocity. When $E_3 = E_{3,max}$, particles 1 and 2 move in the same direction and $p_{\|}^* = p^*$. Thus, the upper bound is given by $E_{2,max} = \gamma\,(E^* + \beta p^* c)$. When $E_3 = m_3 c^2$, particle 3 is not moving in the Y system, and thus particles 1 and 2 move in the opposite directions to conserve the 3-momentum. Then $p_{\|}^* = -p^*$ and $E_{2,min} = \gamma\,(E^* - \beta p^* c)$. Therefore,

$$\gamma\,(E^* - \beta p^* c) \leq E_2 \leq \gamma\,(E^* + \beta p^* c) \,.$$

Problem 9

Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $P^\mu + P_e^\mu = (P')^\mu + (P_e')^\mu$*. [Hint: we would like to eliminate the final elecron 4-momentum* $(P_e')^\mu$*, so make this the subject of the equation and square.]*

A collimated beam of X-rays of energy 17.52 keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of 90°*, including a quantitative indication of the scale.*

In the lab frame, the original electron is stationary, so $P_e = (m_e c, 0)$, whereas for the photon $P = (E_\gamma/c, \vec{p}_\gamma)$, where $E_\gamma = \hbar\omega$, and $|\vec{p}_\gamma| = E_\gamma/c$, with $P^2 = 0$. The 4-vector conservation law reads

$$P + P_e = P' + P_e' \,.$$

It is convenient to eliminate the final electron 4-momentum by writing

$$\left(P + P_e - P'\right)^2 = P_e'^2 = -m_e^2 c^2 \,.$$

Simplifying, we find (recall that $P^2 = 0$ and $P'^2 = 0$ since photons are massless)

$$PP' - PP_e + P'P_e = 0$$

or, explicitly,

$$-\frac{E_\gamma E_\gamma'}{c^2} + \frac{E_\gamma E_\gamma'}{c^2}\cos\varphi = m_e\left(E_\gamma' - E_\gamma\right) \,,$$

where φ is the angle between $\vec{p}_\gamma$ and $\vec{p}_\gamma'$. With $E_\gamma = \hbar\omega = h\nu = hc/\lambda$, we get

$$\lambda' - \lambda = \frac{h}{m_e c}\left(1 - \cos\varphi\right) = \lambda_C\left(1 - \cos\varphi\right) \,,$$

where $\lambda_C \equiv h/m_e c = hc/m_e c^2$ is the Compton wavelength.

We have $hc \approx 1.24 \cdot 10^{-6}\, eV \cdot m$, and the electron's Compton wavelength $\lambda_C \approx 2.43 \cdot 10^{-12}\, m = 0.00243\, nm$. The beam's photons have $\lambda = hc/E_\gamma \approx 0.0708\, nm$. At 90°, the scattered photons will have a peak around $\lambda' = \lambda + \lambda_C \approx 0.0732\, nm$. We may also expect contributions to the spectrum from the photons of lower energy resulting from multiple, rather than single, scattering. A typical spectrum is shown in Fig. 1.

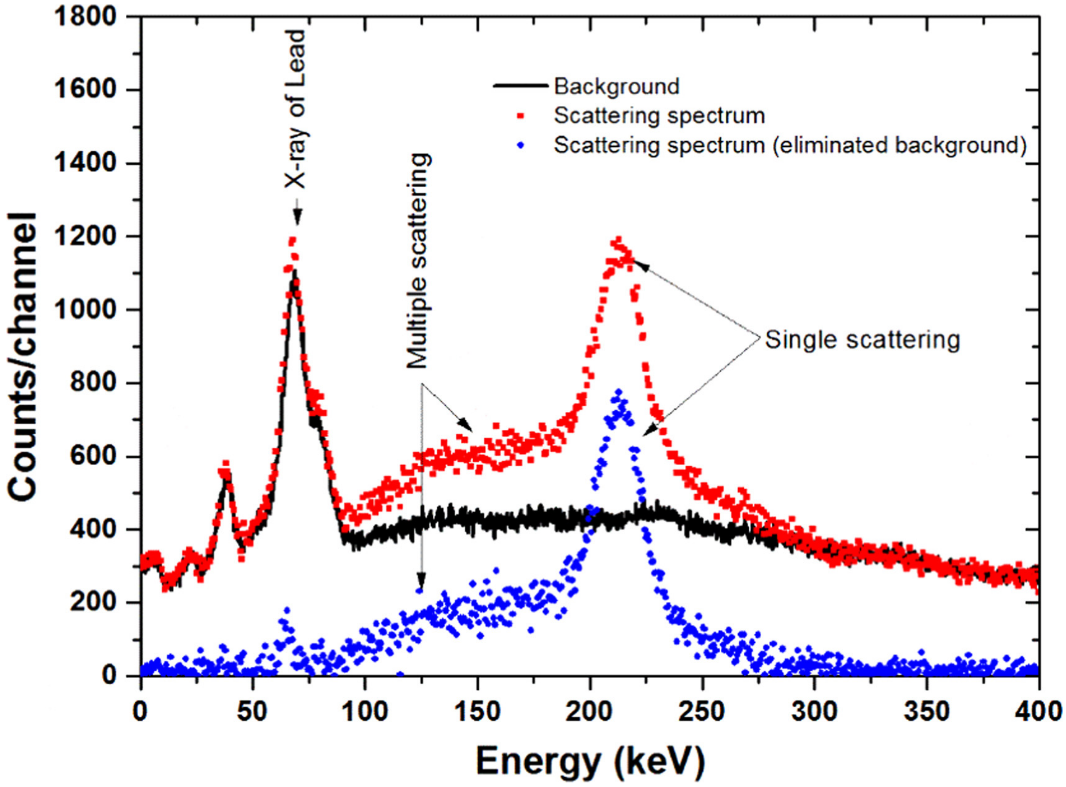


FIG. 1: A typical spectrum of photons after Compton scattering by a target (in this case, a 662 keV photon beam from ^{137}Cs radioactive source was scattered on a steel target at a scattering angle of 120°). Ignore the background and focus on blue dots only. The Compton peak is well visible at the energy of about 225 keV (predicted by the Compton's formula). At lower energies, there is a broad distribution of photons resulting from multiple scattering. Figure from the paper by Tran Thien Thanh et al., "Verification of Compton scattering spectrum of a 662 keV photon beam scattered on a cylindrical steel target using MCNP5 code" in "Applied Radiation and Isotopes", Volume 105, November 2015, Pages 294-298; https://doi.org/10.1016/j.apradiso.2015.09.005.