OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

SYMMETRY AND RELATIVITY

TUTORIAL II

Kinematics and dynamics

Problem Set 2

(Part A: problems 1-4)

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Problem 1

Show, using algebra, a spacetime diagram, or otherwise, that

(i) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval;

(ii) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval;

(iii) for any time-like vector there exists a frame in which its spatial part is zero;

(iv) any vector orthogonal to a time-like vector must be space-like;

(v) with one exception, any vector orthogonal to a null vector is spacelike, and describe the exception;

(vi) the instantaneous 4-velocity of a particle is parallel to the worldline to the worldline (i.e. demonstrate that you understand the meaning of this claim - if you do then it is obvious);

(vii) if the 4-displacement between any two events is orthogonal to an observer's worldline, then the events are simultaneous in the rest frame of that observer.

Solution:

(i) In in some inertial reference frame (IRF) S, consider events A and B with coordinates, repsectively, (ct_A, x_A) and (ct_B, x_B), and let $\Delta x^0 = c(t_B - t_A)$, $\Delta x = x_B - x_A$. If $\Delta x^0 > 0$ and $\Delta x'^0 > 0$ in any other IRF, we can find among them the IRF where $\Delta x' = 0$ (and thus the interval between the events A and B, $s'^2 = -(\Delta x'^0)^2 < 0$, is time-like in this and thus any other IRF). Indeed, using Lorentz transformations, we have $\Delta x' = \gamma(\Delta x - \beta \Delta x^0)$, so choosing $\beta = \frac{\Delta x}{\Delta x^0}$ we get $\Delta x' = 0$. Note that the condition $\Delta x'^0 = \gamma(\Delta x^0 - \beta \Delta x) > 0$ implies $\Delta x^0 > \beta \Delta x$, and therefore $|\beta| = \frac{|\Delta x|}{\Delta x^0} < \frac{1}{|\beta|} \Rightarrow |\beta| < 1$, so the choice is legitimate.

Conversely, if the interval is time-like, does $\Delta x^0 > 0$ imply $\Delta x'^0 > 0$? Consider the time-like interval $s^2_{AB} = -(\Delta x^0)^2 + \Delta x^2 < 0$, so

$$s^2_{AB} = \begin{cases} \Delta x - \Delta x^0 < 0 \quad (\Delta x^0 > \Delta x) \\ \Delta x + \Delta x^0 > 0 \quad (\Delta x^0 > -\Delta x), \end{cases} \tag{1}$$

or

$$\begin{cases} \Delta x - \Delta x^0 > 0 \quad (\Delta x^0 < \Delta x) \\ \Delta x + \Delta x^0 > 0 \quad (\Delta x^0 < -\Delta x), \end{cases} \tag{2}$$

It is convenient to represent this on a diagram (see Fig. 1).

Since $\Delta x'^0 = \gamma(\Delta x^0 - \beta \Delta x)$, $\Delta x^0 > 0$ would imply $\Delta x'^0 > 0$ if $\Delta x^0 > \beta \Delta x$. If β and Δx have

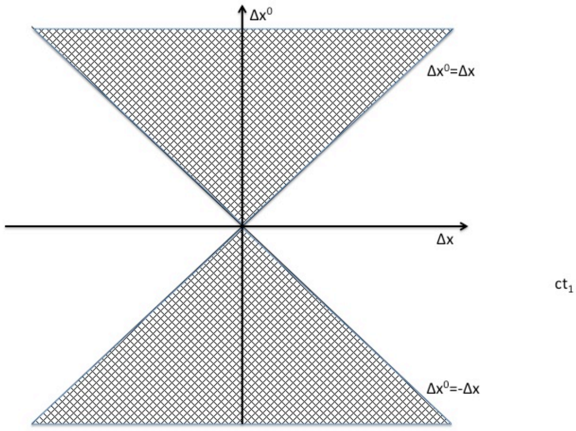


FIG. 1: Space-time diagram.

different signs, the last inequality is obviously true. If they are both positive ($\beta > 0$, $\Delta x > 0$), then this is true because $s^2_{AB} < 0$ implies $\Delta x^0 > \Delta x$ and thus $\Delta x^0 > \Delta x > \beta \Delta x$, since $\beta < 1$.

For $\beta < 0$, $\Delta x < 0$ we can use $\Delta x^0 > -\Delta x = |\Delta x| > |\beta||\Delta x| = \beta \Delta x$, since $|\beta| < 1$. So indeed $\Delta x^0 > \beta \Delta x$ and thus $\Delta x'^0 > 0$.

Thus the notion of *causality* is meaningful for events separated by time-like intervals: one can time-order them, and the ordering is the same in all IRFs.

Note (optional): The parameter space of the general Lorentz group (the group preserving the Minkowski metric) consists of 4 components, depending on the sign of det $\Lambda = \pm 1$ *and on the sign of* Λ^0_0 *(we can have either* $\Lambda^0_0 \geq 1$ *or* $\Lambda^0_0 \leq -1$*). The component with det* $\Lambda = 1$ *and* $\Lambda^0_0 \geq 1$ *contains the identity transformation. This subgroup of the full Lorentz group is called the group of proper orthochronous Lorentz transformations and is often denoted by* $L^{\uparrow}_{+}$*. Other 3 components are denoted* $L^{\uparrow}_{-}$, $L^{\downarrow}_{+}$ *and* $L^{\downarrow}_{-}$*, where* $\pm$ *corresponds to the sign of the determinant. Generic Lorentz transformations are combinations of transformations from* $L^{\uparrow}_{+}$ *and one of the 4 elements* (E, P, T, PT)*, where* $E = diag(1, 1, 1, 1)$ *is the identity transformation,* $P = diag(1, -1, -1, -1)$ *is parity, and* $T = diag(-1, 1, 1, 1)$ *is time reversal.*

One can show that the sign of the component A^0 *of a generic time-llike 4-vector* A^μ *is preserved by the Lorentz transformations* $\Lambda \in L^{\uparrow}$ *(orthochronous Lorentz transformations). Indeed, we have* $A'^\mu = \Lambda^\mu_{\ \nu} A^\nu$ *and* $A'^0 = \Lambda^0_{\ 0} A^0 + \Lambda^0_{\ i} A^i$*, where* $\Lambda^0_{\ 0} > 0$ *since* $\Lambda \in L^{\uparrow}$*. Thus, to show that the sign of* A^0 *remains unchanged, and e.g.* $A^0 > 0$ *implies* $A'^0 > 0$*, we need to show that* $|\Lambda^0_{\ i} A^i| < |\Lambda^0_{\ 0} A^0|$ *for a time-like vector* A^μ*.To estimate* $|\Lambda^0_{\ i} A^i|$*, we use Cauchy-Bunyakovsky inequality:* $(\Lambda^0_{\ i} A^i)^2 \leq (\Lambda^0_{\ i} \Lambda^0_{\ i})(A_i A_i)$*. Since* A^μ *is time-like,* $(A^0)^2 > A_i A_i$*. Also, from* $\Lambda^\mu_{\ \rho} \Lambda^\nu_{\ \sigma} \eta^{\rho\sigma} = \eta^{\mu\nu}$ *we have* $(\Lambda^0_{\ 0})^2 -$

$(\Lambda^0_i)^2 = 1$. *Therefore,* $(\Lambda^0_i A^i)^2 \leq ((\Lambda^0_0)^2 - 1)(A_i A_i) < ((\Lambda^0_0)^2 - 1)(A^0)^2 < (\Lambda^0_0)^2 (A^0)^2$. *So, indeed,* $|\Lambda^0_i A^i| < |\Lambda^0_0 A^0|$ *for a time-like vector* A^μ*, and the sign of the temporal component of* A^μ *remains unchanged.*

(ii) If two events are simultaneous in a certain IRF, the corresponding interval $s^2_{AB} = -c^2\Delta t^2 + \Delta x^2 = \Delta x^2 > 0$ is space-like. If the interval is space-like, $\Delta x^2 - \Delta(x^0)^2 > 0 \Longrightarrow |\Delta x| > |\Delta x^0|$, one can choose IRF such that $\Delta x'^0 = \gamma(\Delta x^0 - \beta \Delta x) = 0$: $|\beta| = \frac{|\Delta x^0|}{|\Delta x|} < 1$.

(iii) We use notation, $A^\mu = (A^0, \vec{A})$. In a time-like case:

$$A \cdot A = \eta_{\mu\nu} A^\mu A^\nu = A_\mu A^\mu = -(A^0)^2 + \vec{A}^2 < 0, \tag{3}$$

(note that this statement is independent of IRF)

$$\begin{aligned} A'^0 &= \gamma(A^0 - \beta A^1) \\ A'^1 &= \gamma(A^1 - \beta A^0) = 0 \Rightarrow \beta = \frac{A^1}{A^0},\ |\beta| = \frac{|A^1|}{|A^0|} < 1, \end{aligned} \tag{4}$$

since $A \cdot A < 0 \Rightarrow |\vec{A}| < |A^0|$.

(iv) $A \cdot B = 0$, $-(A^0)^2 + \vec{A}^2 < 0$ (time-like). Is B space-like (i.e. is it true that $-(B^0)^2 + \vec{B}^2 > 0$ or $|\vec{B}|/|B^0| > 1$)?

Yes, since $A \cdot B = 0$ implies

$$-A^0 B^0 + \vec{A} \cdot \vec{B} = 0 \Rightarrow |A^0||B^0| = |\vec{A}||\vec{B}||\cos\alpha| \Rightarrow \frac{|\vec{B}|}{|B^0|} = \frac{|A^0|}{|\vec{A}||\cos\alpha|} > 1. \tag{5}$$

(v) If A is null, $-(A^0)^2 + \vec{A}^2 = 0$. Then we have the condition $A \cdot B = 0$, so together we have $|A^0| = |\vec{A}|$ and $A \cdot B = 0$. Then

$$A^0 B^0 = \vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\alpha \tag{6}$$

and hence

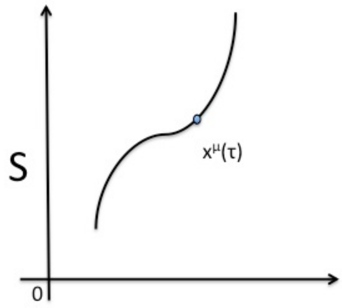
$$\frac{|\vec{B}|}{|B^0|} = \frac{|A^0|}{|\vec{A}||\cos\alpha|} = \frac{1}{|\cos\alpha|} \geq 1, \tag{7}$$

which means that B is space-like except when $\vec{A} \| \vec{B}$ in which case B is also null.

(vi) Worldlines in 3+1-dimensional Minkowski space-time: $x^\mu = x^\mu(\tau)$, where τ is a parameter (e.g. proper time). Then $u^\mu = \frac{dx^\mu}{d\tau}$, i.e. u^μ is tangent to the curve in the usual sense,

(recall $y = y_0 + y'(x)(x - x_0)$ for $y = y(x)$). Now, $ds^2 = -c^2 dt^2 + d\vec{x}^2 = -c^2 d\tau^2$, where $-c^2 dt^2 + d\vec{x}^2$ is describes ds^2 in S and $-c^2 d\tau^2$ in particle's own frame. Therefore

$$d\tau = dt\sqrt{1 - \vec{v}^2/c^2} = \frac{dt}{\gamma}, \tag{8}$$



so that $u^\mu = \frac{dx^\mu}{d\tau} = \left(c\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) = (\gamma c, \gamma\vec{v})$.

(vii) Suppose y_1 and y_2 are two events in S, and $\Delta y^\mu = y_2^\mu - y_1^\mu$ is the 4-displacement. If for an observer with a worldlike $x^\mu = x^\mu(\tau)$ and 4-velocity $u^\mu = \frac{dx^\mu}{d\tau}$ we have

$$\Delta y^\mu u_\mu = -\gamma c^2(t_2 - t_1) + \Delta y^i \gamma v^i = 0 \tag{9}$$

(since $\Delta y^\mu u_\mu$ is Lorentz-invariant, this is true in any IRF), then in the rest frame of the observer (where $\vec{v}' = 0$), $\Delta y'^\mu u'_\mu = -\gamma c^2(t'_2 - t'_1) = 0 \Rightarrow t'_2 = t'_1$, and hence events are simultaneous in that frame.

Problem 2

Define proper time. A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by $X^\mu = (ct, x, y, z)$ in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$. Two particles have 3-velocities $\vec{u}$, $\vec{v}$ in some reference frame. The Lorentz factor for their relative velocity $\vec{w}$ is given by

$$\gamma(w) = \gamma(u)\gamma(v)\left(1 - \vec{u}\cdot\vec{v}/c^2\right). \tag{10}$$

Prove this twice, by using each of the following two methods:

(i) In the given frame, the worldline of the first particle is $X^\mu = (ct, \vec{u}t)$ **See comments in the solution.** *Transform to the rest frame of the other particle to obtain*

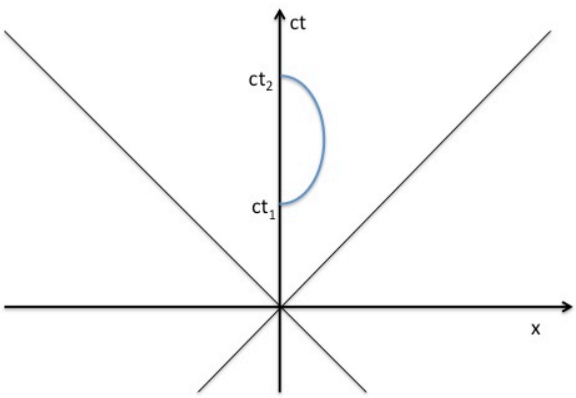
$$t' = \gamma_v t\left(1 - \vec{u}\cdot\vec{v}/c^2\right).$$

Obtain dt'/dt and apply the result of the first part of this question.

(ii) Use the invariant $U^\mu V_\mu$, first showing that it is equal to $-c^2\gamma(w)$.

Solution:

A proper time of an object (e.g. a point particle) is the time in the reference frame where this object is not moving, i.e. where its 3-velocity is zero. In such a frame, the object's 4-coordinate is $X_0^\mu = (c\tau, 0, 0, 0)$, and the infinitesimal interval is $ds^2 = -c^2 d\tau^2$. The same object considered in

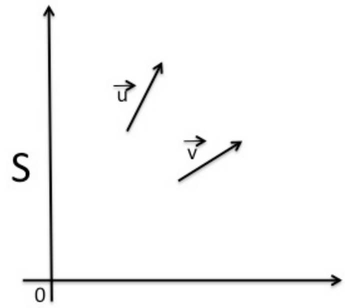


an inertial reference frame has the 4-coordinate $X^\mu = (ct, x(t), y(t), z(t))$. Since the interval is an invariant, we find $ds^2 = -c^2dt^2 + d\vec{x}^2 = -c^2 d\tau^2$ - see problem 1 (vi), with the final equation (8). Now the proper time is $d\tau = dt\sqrt{1 - \vec{v}^2/c^2}$, so

$$\tau = \int_{t_1}^{t_2} dt\sqrt{1 - \vec{v}^2(t)/c^2}. \tag{11}$$

Equation (11) essentially computes the length of a curve in $3+1$ dimensional Minkowski space-time. Note that the straight line path is longer than the one along a curve because the geometry is non-Euclidean.

Now consider the second part of the problem:



(i) In the original reference frame S, the position of a particle with *constant* 3-velocity $\vec{u}$ is described by the 4-vector $X^\mu = (ct, \vec{u}t) = (ct, \vec{r})$. A subtlety here is that in principle there is no need to restrict ourselves to considering constant $\vec{u}$ and $\vec{v}$ (the result for relative velocity is valid regardless of this assumption). If velocities of the particles $\vec{u}$ and $\vec{v}$ are not constant, then $X^\mu = (ct, \vec{u}t)$ is the 4-vector of an instantaneous IRF (and similarly for the particle $\vec{v}$), and thus all transformations below are done between IRFs moving with *constant* velocities $\vec{u}$ and $\vec{v}$ (therefore, using Lorentz transformations is a legitimate step).

Let S' be the inertial reference frame associated with the particle moving with velocity $\vec{v}$ in S. If $\vec{v}$ were parallel to e.g. the x-axis, one could use the standard Lorentz transformations from S to S' :

$$ct' = \gamma\left(ct - \beta x\right) \,, \tag{12}$$

$$x' = \gamma\left(x - \beta ct\right) \,. \tag{13}$$

Note a useful mnemonic rule: on the right-hand-side of Lorentz transformations you can have EITHER PRIME OR MINUS. Here, the velocity $\vec{v}$ is not parallel to the x axis, but we can reduce the current situation to the old one. Indeed, the radius-vector $\vec{r}$ can be split into components parallel and perpendicular to $\vec{v}$: $\vec{r} = \vec{r}_{\parallel} + \vec{r}_{\perp}$, where $\vec{r}_{\parallel} = (\vec{r}\cdot\frac{\vec{v}}{v})\cdot\frac{\vec{v}}{v}$, $\vec{r}_{\perp} = \vec{r} - \vec{r}_{\parallel}$. Now, $|\vec{r}_{\parallel}|$ is the analog of x in eqs (12)-(13). Transforming to S' associated with particle v, we thus get

$$\begin{aligned} ct' &= \gamma(v)\left(ct - \frac{v}{c} r_{\parallel}\right) = \gamma(v)\left(ct - \frac{\vec{r}\cdot\vec{v}}{c}\right) \\ \Rightarrow t' &= \gamma(v)\left(t - \frac{\vec{r}\cdot\vec{v}}{c^2}\right) = \gamma(v)\left(t - \frac{t}{c^2}\vec{u}\cdot\vec{v}\right) = t\gamma(v)\left(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\right) . \end{aligned} \tag{14}$$

In S', the 4-velocity of particle u is $u'^\mu = \frac{dX'^\mu}{d\tau} = (c\gamma(w), \gamma w^i)$, where w^i is the relative velocity (of u w.r.t. v). Therefore $c\gamma(w) = c\frac{dt'}{d\tau}$, so

$$dt' = dt\gamma(v)\left(1 - \frac{\vec{u}\cdot\vec{v}}{c^2}\right)$$

$$-c^2 d\tau^2 = -c^2 dt^2 + \vec{u}^2 dt^2 \Rightarrow d\tau = dt/\gamma(u)$$

$$\Rightarrow \gamma(w) = \gamma(u)\gamma(v)\left(1 - \vec{u}\cdot\vec{v}/c^2\right). \tag{15}$$

(ii) The same result can be obtained by computing the invariant $U^\mu V_\mu$ in different reference rames.

In S, $U^\mu = \frac{dX_1^\mu}{d\tau_1}$ and $V^\mu = \frac{dX_2^\mu}{d\tau_2}$, where X_1^μ and X_2^μ are the 4-coordinates of the two particles, respectively. Explicitly, since $d\tau_1 = dt/\gamma(u)$, $d\tau_2 = dt/\gamma(v)$, we find in S: $U^\mu = (c\gamma(u), \gamma(u)\vec{u})$ and $V^\mu = (c\gamma(v), \gamma(v)\vec{v})$. Therefore

$$U^\mu V_\mu \equiv \eta_{\mu\nu}U^\mu V^\nu = -c^2\gamma(u)\gamma(v) + \gamma(u)\gamma(v)\vec{u}\cdot\vec{v} = -c^2\gamma(u)\gamma(v)\left(1 - \vec{u}\cdot\vec{v}/c^2\right). \tag{16}$$

In S', on the other hand, we have: $V'^\mu = (c, 0, 0, 0)$ and $U'^\mu = (c\gamma(w), \gamma(w)\vec{w})$, where $\vec{w}$ is the velocity of particle u relative to particle v. Therefore,

$$U'^\mu V'_\mu = -c^2\gamma(w). \tag{17}$$

But the product of 4-vectors is Lorentz-invariant, $U'^\mu V'_\mu = U^\mu V_\mu$, so

$$\gamma(w) = \gamma(u)\gamma(v)\left(1 - \vec{u}\cdot\vec{v}/c^2\right), \tag{18}$$

as before. *Note: this result is symmetric in $\vec{u}$ and $\vec{v}$, and insensitive to whether velocities $\vec{u}$ and $\vec{v}$ are constant. The notion of relative velocity is important in considering scattering of relativistic particles (see e.g. [1], section 12). The expression for w is sometimes written as*

$$w^2 = \frac{(\vec{u}-\vec{v})^2 - [\vec{u}\times\vec{v}]^2}{(1 - \vec{u}\cdot\vec{v}/c^2)^2}. \tag{19}$$

For two particles with 4-momenta p_1^μ and p_2^μ in some intertial frame S, $p_{1\mu}p_2^\mu = -\gamma(v_1)\gamma(v_2)m_1m_2c^2 + \gamma(v_1)\gamma(v_2)\vec{v_1}\vec{v_2}$ is an invariant (again, note that $\vec{v_1}$ and $\vec{v_2}$ are not assumed to be constant). In the frame where one of the particles is at rest, $p_{1\mu}p_2^\mu = -\gamma(w)m_1m_2c^2$. This is yet another way to show the validity of Eq. (18) *or Eq.* (19) *(see [1], section 12).*

Comment: Considering relative velocity of two points moving with infinitesimally close velocities $\vec{v}$ and $\vec{v} + d\vec{v}$, we find from Eq. (19)

$$dw^2 = \frac{(c^2 - v^2)(d\vec{v})^2 + (\vec{v}\cdot d\vec{v})^2}{(c^2 - v^2)^2}. \tag{20}$$

In the space with coordinates v_x, v_y, v_z, Eq. (20) defines a metric which is known in mathematics as Lobachevsky metric in three-dimensional non-Euclidean (hyperbolic) space. Thus, the space of velocities in Special Relativity has the geometry of Lobachevsky (hyperbolic) space. In the limit $v/c \to 0$, we return to the Euclidean velocity space $dw^2 \to (d\vec{v})^2 = dv_x^2 + dv_y^2 + dv_z^2$.

Problem 3

Derive a formula for the frequency ω of light waves from a moving source, in terms of the proper frequency ω_0 in the source frame and the angle in the observer's frame, θ, between the direction of observation (we take it to be the direction from the source to the observer, otherwise it is $\pi - \theta$) and the velocity of the source.

A galaxy with a negligible speed of recession from Earth has an active nucleus. It has emitted two jets of hot material with the same speed v in opposite directions, at an angle θ to the direction to Earth. A spectral line in singly-ionised Mg (proper wavelength $\lambda_0 = 448.1$ nm) is emitted from both jets. Show that the wavelengths $\lambda_\pm$ observed on Earth from the two jets are given by

$$\lambda_\pm = \lambda_0 \gamma \left(1 \pm \frac{v}{c}\cos\theta\right) \tag{21}$$

(you may assume the angle subtended at Earth by the jets is negligible). If $\lambda_+ = 420.2$ nm and $\lambda_- = 700.1$ nm, find v and θ. In some cases, the receding source is difficult to observe. Suggest a reason for this.

Solution:

The most convenient way to derive the relativistic Doppler shift formula is to use the invariants.

A source moving w.r.t. an inertial frame S (e.g. w.r.t. Earth) has 4-velocity $V^\mu = (\gamma c, \gamma \vec{v})$ in that frame and 4-velocity $V'^\mu = (c, 0, 0, 0)$ in the frame S' associated with the source. A photon emitted by the sourse has 4-momentum $k^\mu = (\frac{\hbar\omega}{c}, \hbar\vec{k})$ in S (we shall set $\hbar = 1$ in the following). Since photon is massless, we have $k^2 = 0$, and thus $|\vec{k}| = \omega/c$. In S', $k'^\mu = (\frac{\omega_0}{c}, \vec{k}')$. Since the product $k^\mu V_\mu$ is an invariant, we have

$$k'^\mu V'_\mu = k^\mu V_\mu \,, \tag{22}$$

or, explicitly,

$$-\omega_0 = -\gamma\omega + \gamma\,\vec{v}\cdot\vec{k} = -\gamma\omega + \gamma\,|\vec{v}||\vec{k}|\cos\theta\,, \tag{23}$$

where θ is the angle between the direction of $\vec{v}$ and the direction of $\vec{k}$. This gives

$$\boxed{\omega = \frac{\omega_0\sqrt{1-\beta^2}}{1-\beta\cos\theta}\,.} \tag{24}$$

Using $\lambda\nu = c$ and $\omega = 2\pi\nu$, we find the formula for the wavelength

$$\boxed{\lambda = \lambda_0\gamma\,(1-\beta\cos\theta)\,.} \tag{25}$$

Assuming that $\vec{k}$ is directed towards the Earth, one can see that for $\theta = 0$ (the source is moving directly towards the Earth), $\lambda < \lambda_0$ (the line is blue-shifted), whereas for $\theta = \pi$ (the source is moving directly away from the Earth), $\lambda > \lambda_0$ (the line is red-shifted). There exists an angle θ_* for which neither blue- nor red-shift is observed ($\lambda = \lambda_0$). From eq. (25) one can see that

$$\cos\theta_* = \frac{c}{v}\left(1 - \sqrt{1 - v^2/c^2}\right) \approx \frac{v}{2c} \qquad \text{for } \frac{v}{c} \ll 1\,. \tag{26}$$

Thus, for $v/c \ll 1$, $\theta_* \approx \pi/2$.

In the scenario with the galaxy emitting two jets, for the jet moving towards the Earth, we have

$$\lambda = \lambda_- = \lambda_0\gamma\left(1 - \beta\cos\theta\right)\,, \tag{27}$$

whereas for the jet moving away from the Earth

$$\lambda = \lambda_+ = \lambda_0\gamma\left[1 - \beta\cos\left(\pi - \theta\right)\right] = \lambda_0\gamma\left(1 + \beta\cos\theta\right)\,. \tag{28}$$

Substituting numbers, one finds $v \approx 0.6c$ and $\theta \approx 65.4\,^\circ$.

The intensity of the radiation in S requires separate consideration, but one can show (see e.g. [2] or Chapter 3 of [3]) that for sources, moving towards the Earth, $I > I_0$, whereas for the ones receding we have $I < I_0$. Thus, the receding source will be dimmer. More precisely, the flux of energy I (energy per time per solid angle) in S in related to the flux I_0 in the frame S' of the source by

$$I = \left(\frac{\omega}{\omega_0}\right)^4 I_0 = \frac{(1-\beta^2)^2}{(1-\beta\cos\theta)^4}\, I_0\,. \tag{29}$$

In particular, for the sources directly approaching the observer and directly receding from him/her, we have

$$\frac{I_{approaching}}{I_{receding}} = \left(\frac{1+\beta}{1-\beta}\right)^4 > 1\,. \tag{30}$$

Comments:

- It has not been specified in the problem whether the velocity of the source $\vec{v}$ is constant. If it is not constant, what is meant by $\vec{v}$ is the instantaneous velocity (in frame S) of the source at the moment (again, in S) the photon was emitted. The presented solution (which is based on the invariance of the scalar product of 4-vectors) is insensitive to this, and the result (24) remains valid.

  But the result (24) can also be obtained by applying Lorentz transformation to the photon's 4-momentum $k^\mu = (\omega/c, \vec{k})$ is S:

$$k'^0 = \gamma(k^0 - \vec{\beta}\cdot\vec{k})\,, \tag{31}$$

where $\vec{\beta} = \vec{v}/c$ is the (normalised) velocity of the source in S. However, if $\vec{v}$ is not constant, the frame associated with the source is not an inertial frame, and Lorentz transformations apparently cannot be applied. They can be applied, nevertheless, to the *instantaneous inertial reference frame S' comoving with the source* (i.e. the frame whose velocity is contant and coincides with the velocity of the source at the moment when the photon is emitted). Since $k'^0 = \omega_0/c$, where ω_0 is the frequency of the source in the frame where its velocity is zero (i.e. S'), Eq. (31) gives the same result as (24). Similar considerations apply in the situation when the source is stationary in an inertial frame but the observer is moving with non-constant velocity, as in the experiment described below.

A simple experiment confirming the result (31) (and, indirectly, the concept of the instantaneous comoving inertial reference frame) was performed in 1960 by Hay, Schiffer, Cranshaw and Engelstaff. The source of photons was located in the centre of a rotating disk of radius R, and the detector on the rim (thus the detector was moving with acceleration). Since in the lab frame S in this case $\vec{\beta}\cdot\vec{k} = 0$, where $\vec{\beta}$ is the linear velocity of the rim, the frequency detected on the rim should be $\omega' = \gamma\omega$, where ω is the frequency of the source in the centre of the disk.

One may be legitimately concerned about using formulas such as (24) in situations where the velocity v is not constant, since the process of emitting a photon is not exactly instantaneous (the relevant time-scale is $\Delta t = T = 2\pi/\omega = 1/\nu$) and v may change appreciably during that time interval [4]. During the time $\Delta t = T$ which it takes for a complete wave with the frequency $\nu = 1/T$ to pass a point in the observer's comoving instantaneous rest frame, the observer, moving with acceleration a, would move by a distance $a\Delta t^2/2 = a/2\nu^2$. This distance better be small in comparison with the wavelength $\lambda = c/\nu$, i.e. we should have $a \ll 2c\nu$ in order to be able to replace the accelerated observer frame wih the comoving instantaneous inertial one [4]. For e.g. yellow light, this is $a \ll 3\cdot 10^{26} g$ (in the 1960 experiment mentioned above, the acceleration was $a \sim 6\cdot 10^4 g$).

A careful discussion of these issues can be found in Rindler's books [5] and [4].

- Another remark is that the emitted waves may not necessarily be electromagnetic waves. For sound waves, $k^\mu = (\omega/c, \vec{k})$, where $\omega = v_s|\vec{k}|$ for small $|\vec{k}|$. Correspondingly, Eq. (23) reads

$$-\omega_0 = -\gamma\omega + \gamma\,\vec{v}\cdot\vec{k} = -\gamma\omega + \gamma\,|\vec{v}||\vec{k}|\cos\theta = -\gamma\omega + \gamma\,|\vec{v}|\frac{\omega}{v_s}\cos\theta\,, \tag{32}$$

This gives the following formula for relativistic acoustic Doppler effect (here $v = |\vec{v}|$)

$$\boxed{\omega = \frac{\omega_0 \sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v}{v_s} \cos\theta}\,.} \tag{33}$$

In the non-relativistic limit, the relativistic square root is gone and we have

$$\omega = \frac{\omega_0}{1 - \frac{v}{v_s} \cos\theta} \approx \omega_0 \left(1 + \frac{v}{v_s} \cos\theta\right) \tag{34}$$

for $v/v_s \ll 1$.

Problem 4

The 4-angular momentum of a single particle about the origin is defined as

$$L^{\mu\nu} \equiv X^\mu P^\nu - X^\nu P^\mu \, .$$

(i) Prove that in the absence of forces, $dL^{\mu\nu}/d\tau = 0$.

(ii) Exhibit the relationship between the space-space part L^{ij} and the 3-angular momentum vector $\mathbf{L} = \mathbf{x} \times \mathbf{p}$.

(iii) The total angular momentum of a collection of particles about the pivot R^λ is defined as

$$L^{\mu\nu}_{\text{tot}} = \sum_i \left(X^\mu_i - R^\mu\right) P^\nu_i - \left(X^\nu_i - R^\nu\right) P^\mu_i \, ,$$

where the sum runs over the particles (that is, X^μ and P^μ are 4-vectors, not second-rank tensors; i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

Solution:

(i) We have, generically,

$$\frac{dp^\mu}{d\tau} = \left(\frac{1}{c}\frac{dE}{d\tau}, \frac{d\vec{p}}{d\tau}\right) = \left(\frac{\gamma}{c}\frac{dE}{dt}, \gamma\frac{d\vec{p}}{dt}\right) \, .$$

The absence of forces ($\vec{f} = 0$) implies that $dp^\mu/d\tau = 0$, since $dE/dt = \vec{f} \cdot \vec{v} = 0$ and $d\vec{p}/dt = \vec{f} = 0$. Therefore,

$$\frac{dL^{\mu\nu}}{d\tau} = \frac{dX^\mu}{d\tau}P^\nu - \frac{dX^\nu}{d\tau}P^\mu = U^\mu P^\nu - U^\nu P^\mu = m\left(U^\mu U^\nu - U^\nu U^\mu\right) = 0 \, .$$

(ii) The space-space part L^{ab} (where $a, b = 1, 2, 3$) of $L^{\mu\nu}$ can be written as a 3x3 antisymmetric matrix

$$L^{ab} = \begin{pmatrix} 0 & L^{12} & L^{13} \\ L^{21} & 0 & L^{23} \\ L^{31} & L^{32} & 0 \end{pmatrix} \, ,$$

where $L^{12} = xp_y - yp_x$, $L^{13} = xp_z - zp_x$, $L^{23} = yp_z - zp_y$. On the other hand, $\mathbf{L} = \mathbf{x} \times \mathbf{p} = (L_x, L_y, L_z)$, where the components are determined by the matrix

$$\begin{pmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{pmatrix} \, ,$$

or else by $L_i = \epsilon_{ijk} x_j p_k$. We therefore have $L_x = L^{23}$, $L_y = -L^{13}$, $L_z = L^{12}$, which can also be written as $L_i = \frac{1}{2}\epsilon_{ijk} L^{jk}$.

(iii) The space-space part L^{ab}_{tot} (where $a, b = 1, 2, 3$) of $L^{\mu\nu}_{\text{tot}}$ is

$$\begin{aligned} L^{ab}_{\text{tot}}(R) &= \sum_i \left[(X^a_i - R^a) P^b_i - \left(X^b_i - R^b \right) P^a_i \right] \\ &= \sum_i \left(X^a_i P^b_i - X^b_i P^a_i \right) - R^a \sum_i P^a_i + R^b \sum_i P^b_i \,, \end{aligned} \tag{35}$$

where the summation over i is the summation over the particles. Thus,

$$\sum_i P^a_i = P^a_{\text{tot}} \,, \qquad \text{and} \qquad \sum_i P^b_i = P^b_{\text{tot}} \,.$$

In CMF, $\vec{P}_{\text{tot}} = 0$, i.e. each component P^a_{tot}, $a = 1, 2, 3$, is zero. Thus,

$$L^{ab}_{\text{tot}}(R) = \sum_i \left(X^a_i P^b_i - X^b_i P^a_i \right) = L^{ab}_{\text{tot}}(0) \,.$$

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[1] L. Landau and E. Lifshitz, *The Classical Theory of Fields: Volume 2 (Course of Theoretical Physics)* (Butterworth-Heinemann, 1980).

[2] A. Lightman, W. Press, R. Price, and S. Teukolsky, *Problem Book in Relativity and Gravitation* (Princeton University Press, 2017).

[3] A. Steane, *Relativity Made Relatively Easy* (Oxford University Press, 2012).

[4] W. Rindler, *Relativity: Special, General, and Cosmological* (Oxford University Press, 2006).

[5] W. Rindler, *Introduction to Special Relativity* (Clarendon Press, Oxford, 1991).