B2: Symmetry and Relativity
Problem Set 2: Special Relativity
MT 2019

Conventions: Greek indices take values 0 through 3, while Latin indices take values 1 through 3. 3-vectors can also be indicated by boldface, *e.g.*, $\mathbf{a}$. Minkowski metric $g^{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$.

1. Show, using algebra, a spacetime diagram, or otherwise,

   (i) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval;

   (ii) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval.

   (iii) for any time-like vector there exists a frame in which its spatial part is zero;

   (iv) any vector orthogonal to a time-like vector must be space-like;

   (v) with one exception, any vector orthogonal to a null vector is space-like, and describe the exception.

   (vi) the instantaneous 4-velocity of a particle is parallel to the worldline (*i.e.*, demonstrate that you understand the meaning of this claim—if you do then it is obvious);

   (vii) if the 4-displacement between any two events is orthogonal to an observer's worldline, then the events are simultaneous in the rest frame of that observer.

2. Define *proper time*. A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by $X^\mu = (ct, x, y, z)$ in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$.

   Two particles have 3-velocities $\mathbf{u}$ and $\mathbf{v}$ in some reference frame. The Lorentz factor for their relative 3-velocity $\mathbf{w}$ is given by

   $$\gamma_w = \gamma_u \gamma_v (1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

   Prove this twice, by using each of the following two methods:

   (i) In the given frame, the worldline of the first particle is $X^\mu = (ct, \mathbf{u}t)$. Transform to the rest frame of the other particle to obtain

   $$t' = \gamma_v t(1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

   Obtain dt'/dt and apply the result of the first part of this question.

   (ii) Use the invariant $U^\mu V_\mu$, first showing that it is equal to $-c^2\gamma_w$.

3. Derive a formula for the frequency ω of light waves from a moving source, in terms of the proper frequency ω_0 in the source frame and the angle in the observer's frame, θ, between the direction of observation and the velocity of the source.

   A galaxy with a negligible speed of recession from Earth has an active nucleus. It has emitted two jets of hot material with the same speed v in opposite directions, at an angle θ to the direction to the Earth. A spectral line in singly-ionised Mg (proper wavelength $\lambda_0 = 448.1$ nm) is emitted from both jets. Show that the wavelengths $\lambda_\pm$ observed on Earth from the two jets are given by

$$\lambda_\pm = \lambda_0 \gamma (1 \pm (v/c)\cos\theta)$$

   (you may assume the angle subtended at Earth by the jets is negligible). If $\lambda_+ = 420.2$ nm and $\lambda_- = 700.1$ nm, find v and θ.

   In some cases, the receding source is difficult to observe. Suggest a reason for this.

4. The 4-angular momentum of a single particle about the origin is defined as

$$L^{\mu\nu} \equiv X^\mu P^\nu - X^\nu P^\mu.$$

   (i) Prove that, in the absence of forces, $dL^{\mu\nu}/d\tau = 0$.
   (ii) Exhibit the relationship between the space-space part L^{ij} and the 3-angular momentum vector $\mathbf{L} = \mathbf{x} \wedge \mathbf{p}$.
   (iii) The total angular momentum of a collection of particles about the pivot R^λ is defined as

$$L^{\mu\nu}_{\text{tot}}(R^\lambda) = \sum_i (X^\mu_i - R^\mu)P^\nu_i - (X^\nu_i - R^\nu)P^\mu_i$$

   where the sum runs over the particles (that is, X^μ and P^μ are 4-vectors, not 2nd-rank tensors; i here labels the particles). Show that the 3-angular momentum in the CM frame is independent of the pivot.

5. The 4-vector field F^μ is given by $F^\mu = 2x^\mu + k^\mu(x^\nu x_\nu)$ where k^μ is a constant 4-vector and $x^\mu = (ct, x, y, z)$ is the 4-vector displacement in spacetime. Evaluate the following:

   (i) $\partial_\lambda x^\lambda$
   (ii) $\partial^\mu(x_\lambda x^\lambda)$
   (iii) $\partial^\mu \partial_\mu x^\nu x_\nu$
   (iv) $\partial_\lambda F^\lambda$
   (v) $\partial^\mu(\partial_\lambda F^\lambda)$
   (vi) $\partial^\mu \partial_\mu \sin(k_\lambda x^\lambda)$
   (vii) $\partial^\mu x^\nu$

6. A particle of rest mass m and kinetic energy $3mc^2$ strikes a stationary particle of rest mass $2m$ and combines with it while still conserving energy and momentum. Find the rest mass and speed of the composite particle.

7. Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction, in terms of E_0 and the electron rest mass m.

   High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4} eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

8. A particle Y decays into three other particles, with labels indicated by $Y \to 1 + 2 + 3$. Working throughout in the CM frame:

   (i) Show that the 3-momenta of the decay products are coplanar.

   (ii) Show that the energy of particle 3 is given by

   $$E_3 = \frac{(m_Y^2 + m_3^2 - m_1^2 - m_2^2)c^4 - 2E_1E_2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2}{2m_Y c^2}.$$

   (iii) Show that the maximum value of E_3 is

   $$E_{3,\max} = \frac{m_Y^2 + m_3^2 - (m_1 + m_2)^2}{2m_Y} c^2$$

   and explain under what circumstances this maximum is attained.

   (iv) Show that, when particle 3 has its maximum possible energy, particle 1 has the energy

   $$E_1 = \frac{m_1(m_Y c^2 - E_{3,\max})}{m_1 + m_2}.$$

   [Hint: first argue that 1 and 2 have the same speed in this situation.]

   (v) Now let's return to the more general circumstance, with E_3 not necessarily maximal. Let X be the system composed of particles 1 and 2. Show that its rest mass is given by

   $$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2.$$

   (vi) Write down an expression for the energy E^* of particle 2 in the rest frame of X, in terms of m_1, m_2, and m_X.

   (vii) Show that, when particle 3 has an energy of intermediate size, $m_3c^2 < E_3 < E_{3,\max}$, the energy of particle 2 in the original frame (the rest frame of Y) is in the range

   $$\gamma(E^* - \beta p^* c) \le E_2 \le \gamma(E^* + \beta p^* c)$$

   where E^* and p^* are the energy and momentum of particle 2 in the X frame, and γ and β refer to the speed of that frame relative to the rest frame of Y.

9. Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $P^\mu + P_e^\mu = (P')^\mu + (P_e')^\mu$. [Hint: we would like to eliminate the final electron 4-momentum $(P_e')^\mu$, so make this the subject of the equation and square.] A collimated beam of X-rays of energy 17.52 keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of 90°, including a quantitative indication of the scale.

Additional questions

10. Derive the equations describing the transformation of velocity:

$$\begin{aligned}\mathbf{u}'_{\parallel} &= \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2} \\ \mathbf{u}'_{\perp} &= \frac{\mathbf{u}_{\perp}}{\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)}\end{aligned}$$

11. For any two future-pointing time-like vectors U^μ and V^μ, prove that $U^\mu V_\mu = -uv\cosh\rho$, where ρ is the relative rapidity of frames in which U^μ and V^μ are purely temporal and with $U^\mu U_\mu = -u^2$ and $V^\mu V_\mu = -v^2$.

12. In a given inertial frame S, two particles are shot out from a point in orthogonal directions with equal speeds v. At what rate does the distance between the particles increase in S? What is the speed of each particle relative to the other?

13. In a frame S a guillotine blade in the (x, y) plane falls in the negative y direction towards a block level with the x axis and centred at the origin. The angle of the edge of the blade is such that the point of intersection of blade and block moves at a speed in excess of c in the positive x direction. Show that in some frames S' in standard configuration with S, this point moves in the *opposite* direction along the block. [For simplicity, assume that the blade drops with constant speed u.]

    Now suppose that when the centre of the blade arrives at the block, the whole blade instantaneously evaporates in frame S (for example, it could be vapourized by a very powerful laser beam incident from the z direction). A piece of paper placed on the block is therefore cut on the negative x-axis only. Explain this in S'.

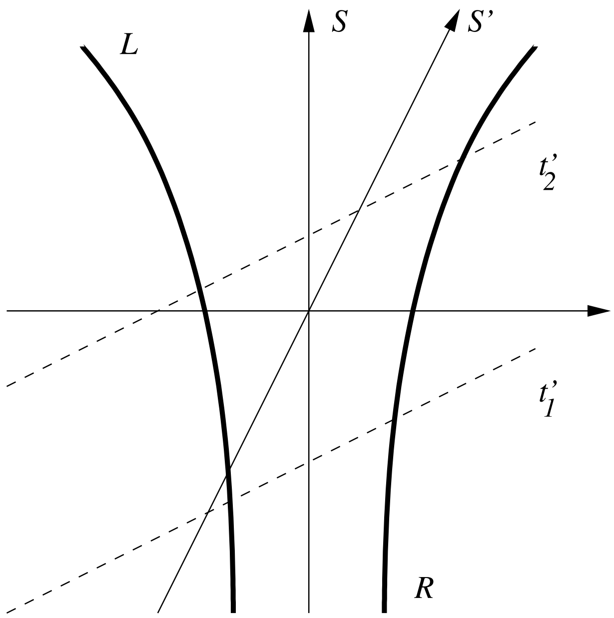
Energy-momentum conservation

14. The upper atmosphere of the Earth receives electromagnetic energy from the sun at the rate 1400 Wm^{-2}. Find the rate of loss of mass of the sun due to all its emitted radiation. [The Earth-sun distance is 499 light seconds.]

15. Calculate the mass increase of a block of copper heated from 0°C to 1000°C, assuming the specific heat capacity is constant at 420 $\mathrm{J\,kg^{-1}K^{-1}}$ and the initial mass is 10 kg.

16. Show that if a 4-vector has a component which is zero in *all* frames, then the entire vector is zero. What insight does this offer into energy and momentum?

17. The following spacetime diagram shows the worldlines of two accelerating particles L and R, and two inertial observers S and S'. The dashed lines are lines of simultaneity in frame S' at two times t'_1 and t'_2.



The two particles have the same mass.

First consider the observations in frame S. At any given time in S, the particles have equal and opposite momenta, therefore their total 3-momentum $\mathbf{p}_{\text{tot}} = \mathbf{p}_L + \mathbf{p}_R = 0$. In other words, $\mathbf{p}_{\text{tot}}$ is constant (and zero).

Now consider the observations in frame S'. Initially the two particles have almost the same velocity relative to S'. Then, between times t'_1 and t'_2, R comes to rest relative to S', while L changes its velocity relative to S' by only a small amount. Therefore the total particle momentum in frame S' roughly halves between times t'_1 and t'_2. The total momentum of the two particles is certainly not constant in S'.

What does the situation tell us about total momentum? Is total momentum a meaningful physical concept? If so, then is it always conserved? Under what conditions does it transform as part of a 4-vector?

18. A system consists of two photons, each of energy E, propagating at right angles in the laboratory frame. Find the rest mass of the system and the velocity of its CM frame relative to the laboratory frame.

Particle formation

19. A particle of mass m and energy E (in the laboratory frame) hits a free stationary target of mass M. If E is greater than a threshold energy E_{th}, the collision produces a number of collision products with masses m_i. Show that E_{th} is given by

$$E_{\text{th}} = \frac{(\sum_i m_i)^2 - m^2 - M^2}{2M} c^2.$$

20. A particle formation experiment creates reactions of the form $b + t \to b + t + n$ where b is an incident particle of mass m, t is a target of mass M at rest in the laboratory frame, and n is a new particle. Define the "efficiency" of the experiment as the ratio of the rest energy of the new particle to the supplied kinetic energy of the incident particle. Show that, at threshold, the efficiency thus defined is equal to

$$\frac{M}{m + M + m_n/2}.$$

tem Pion formation can be achieved by the process $p + p \to p + p + \pi^0$. A proton beam strikes a target containing stationary protons. Calculate the minimum kinetic energy which must be supplied to an incident proton to allow pions to be formed, and compare this to the rest energy of a pion.

21. A photon is incident on a stationary proton. Find, in terms of the rest masses, the threshold energy of the photon if a neutron and a pion are to emerge.

Particle decay

22. Particle tracks are recorded in a bubble chamber subject to a uniform magnetic field of 2 tesla. A vertex consisting of no incoming and two outgoing tracks is observed. The tracks lie in the plane perpendicular to the magnetic field, with radii of curvature 1.67 m and 0.417 m, and separation angle 21°. It is believed that they belong to a proton (mass 939.3 MeV/c^2) and pion (mass 139.6 MeV/c^2), respectively. Assuming this, and that the process at the vertex at the vertex is decay of a neutral particle into two products, find the rest mass of the neutral particle.

23. A particle with known rest mass M and energy E decays into two products with known rest masses m_1 and m_2. Find the energies E_1 and E_2 (in the lab frame) of the products, by the following steps:

    (i) Find the energies E'_1 and E'_2 of the products in the CM frame.

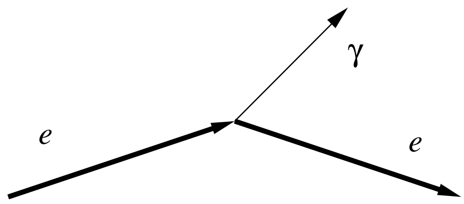
    (ii) Show that the momentum of either decay product in the CM frame is

    $$p = \frac{c}{2M}[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2 m_2^2]^{1/2}.$$

    (iii) Find the Lorentz factor and the speed v of the CM frame relative to the lab.

    (iv) Write down, in terms of v, γ, p, E'_1 and E'_2, expressions for E_1 and E_2 when the products are emitted (1) along the line of flight, and (2) at right angles to the line of flight in the CM frame.

24. This diagram illustrates a process in which an electron emits a photon:



    Prove that the process is impossible. Prove also that a photon cannot transform into an electron-positron pair in free space, and that a photon in free space cannot decay into a pair of photons with differing directions of propagation.

25. A decay mode of the neutral Kaon is $K^0 \to \pi^+ + \pi^-$. The Kaon has momentum 300 MeV/c in the laboratory, and one of the pions is emitted, in the laboratory, in a direction perpendicular to the velocity of the Kaon. Find the momenta of both pions. [Use 497.611 MeV/c^2 as the mass of the kaon, and 139.57061 MeV/c^2 as the mass of the pion.]

Scattering

26. Consider a head-on elastic collision between a moving "bullet" of rest mass m and a stationary target of rest mass M such that the bullet recoils in the opposite of its original direction. Show that the post-collision Lorentz factor γ of the bullet cannot exceed $(m^2 + M^2)/(2mM)$. (This means that for large energies almost all the energy of the bullet is transferred to the target, very different from the classical result.) [Hint: consider $P_t^\mu - Q_b^\mu$, where P_t^μ is the initial 4-momentum of the target and Q_b^μ is the final 4-momentum of the bullet.]

27. Particles of mass m and kinetic energy T are incident on similar particles at rest in the laboratory. Show that, if elastic scattering takes place, then the minimum angle between the final momenta in the laboratory is given by

    $$\cos\theta_{\min} = \frac{T}{T + 4mc^2}.$$