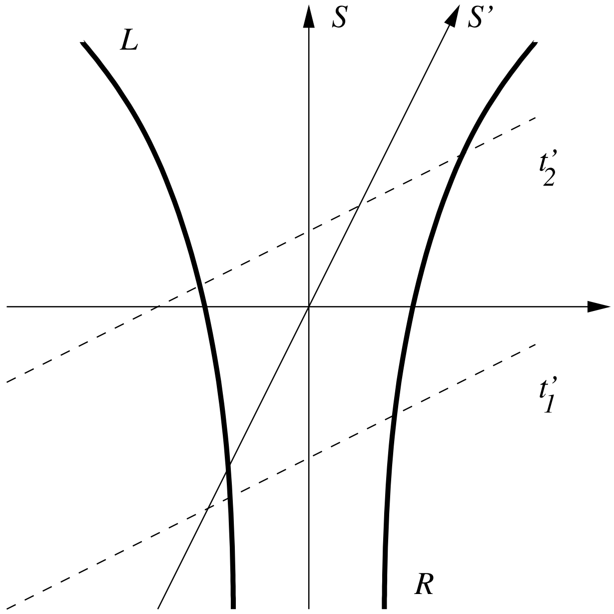
B2: Symmetry and Relativity
Problem Set 3: Momentum and force
MT 2022 Weeks 4-5

1. The following spacetime diagram shows the worldlines of two accelerating particles L and R, and two inertial observers S and S'. The dashed lines are lines of simultaneity in frame S' at two times t'_1 and t'_2.



The two particles have the same mass.

First consider the observations in frame S. At any given time in S, the particles have equal and opposite momenta, therefore their total 3-momentum $\mathbf{p}_{\rm tot} = \mathbf{p}_L + \mathbf{p}_R = 0$. In other words, $\mathbf{p}_{\rm tot}$ is constant (and zero).

Now consider the observations in frame S'. Initially the two particles have almost the same velocity relative to S'. Then, between times t'_1 and t'_2, R comes to rest relative to S', while L changes its velocity relative to S' by only a small amount. Therefore the total particle momentum in frame S' roughly halves between times t'_1 and t'_2. The total momentum of the two particles is certainly not constant in S'.

What does the situation tell us about total momentum? Is total momentum a meaningful physical concept? If so, then is it always conserved? Under what conditions does it transform as part of a 4-vector?

2. A particle of rest mass m and kinetic energy $3mc^2$ strikes a stationary particle of rest mass $2m$ and combines with it while still conserving energy and momentum. Find the rest mass and speed of the composite particle.

3. Two photons may collide to produce an electron-positron pair. If one photon has energy E_0 and the other has energy E, find the threshold value of E for this reaction, in terms of E_0 and the electron rest mass m.

   High energy photons of galactic origin pass through the cosmic microwave background radiation which can be regarded as a gas of photons of energy 2.3×10^{-4} eV. Calculate the threshold energy of the galactic photons for the production of electron-positron pairs.

4. A particle Y decays into three other particles, with labels indicated by $Y \to 1+2+3$. Working throughout in the CM frame:

   (i) Show that the 3-momenta of the decay products are coplanar.

   (ii) Show that the energy of particle 3 is given by

   $$E_3 = \frac{(m_Y^2 + m_3^2 - m_1^2 - m_2^2)c^4 - 2E_1E_2 + 2\mathbf{p}_1 \cdot \mathbf{p}_2 c^2}{2m_Y c^2}.$$

   (iii) Show that the maximum value of E_3 is

   $$E_{3,\text{max}} = \frac{m_Y^2 + m_3^2 - (m_1 + m_2)^2}{2m_Y} c^2$$

   and explain under what circumstances this maximum is attained.

   (iv) Show that, when particle 3 has its maximum possible energy, particle 1 has the energy

   $$E_1 = \frac{m_1(m_Y c^2 - E_{3,\text{max}})}{m_1 + m_2}.$$

   [Hint: first argue that 1 and 2 have the same speed in this situation.]

   (v) Now let's return to the more general circumstance, with E_3 not necessarily maximal. Let X be the system composed of particles 1 and 2. Show that its rest mass is given by

   $$m_X^2 = m_Y^2 + m_3^2 - 2m_Y E_3/c^2.$$

   (vi) Write down an expression for the energy E^* of particle 2 in the rest frame of X, in terms of m_1, m_2, and m_X.

   (vii) Show that, when particle 3 has an energy of intermediate size, $m_3c^2 < E_3 < E_{3,\text{max}}$, the energy of particle 2 in the original frame (the rest frame of Y) is in the range

   $$\gamma(E^* - \beta p^* c) \le E_2 \le \gamma(E^* + \beta p^* c)$$

   where E^* and p^* are the energy and momentum of particle 2 in the X frame, and γ and β refer to the speed of that frame relative to the rest frame of Y.

5. Obtain the formula for the Compton effect using 4-vectors, starting from the usual energy-momentum conservation $P^\mu + P_e^\mu = (P')^\mu + (P_e')^\mu$. [Hint: we would like to eliminate the final electron 4-momentum $(P_e')^\mu$, so make this the subject of the equation and square.] A collimated beam of X-rays of energy 17.52 keV is incident on an amorphous carbon target. Sketch the wavelength spectrum you would expect to be observed at a scattering angle of 90°, including a quantitative indication of the scale.

6. Obtain the transformation equations for 3-force, by starting from the Lorentz transformation of energy-momentum, and then differentiating with respect to t'. [Hint: argue that the relative velocity $\mathbf{v}$ of the reference frame is constant, and use or derive an expression for dt/dt'.]

7. A particle moves hyperbolically with proper acceleration a_0, starting from rest at $t = 0$. At $t = 0$ a photon is emitted towards the particle from a distance c^2/a_0 behind it. Prove that in the instantaneous rest frames of the particle, the distance to the photon is always c^2/a_0.

8. Consider motion under a constant force, for a non-zero initial velocity in an arbitrary direction, as follows:

   (i) Write down the solution for the 3-momentum $\mathbf{p}$ as a function of time, taking as initial condition $\mathbf{p}(0) = \mathbf{p}_0$.

   (ii) Show that the Lorentz factor as a function of time is given by $\gamma^2 = 1 + \alpha^2$ where $\boldsymbol{\alpha} = (\mathbf{p}_0 + \mathbf{f}t)/mc$.

   (iii) You can now write down the solution for $\mathbf{v}$ as a function of time. Do so.

   (iv) Now restrict attention to the case where $\mathbf{p}_0$ is perpendicular to $\mathbf{f}$. Taking the x-direction along $\mathbf{f}$ and the y-direction along $\mathbf{p}_0$, show that the trajectory is given by

$$
\begin{aligned}
x &= \frac{c}{f}(m^2c^2 + p_0^2 + f^2t^2)^{1/2} + \text{const} \\
y &= \frac{cp_0}{f}\log\left(ft + \sqrt{m^2c^2 + p_0^2 + f^2t^2}\right) + \text{const}
\end{aligned}
$$

   where you may quote that $\int (a^2 + t^2)^{-1/2} dt = \log(t + \sqrt{a^2 + t^2})$.

   (v) Explain (without carrying out the calculation) how the general case can then be treated by a suitable Lorentz transformation. [N.B. the calculation as a function of proper time is best done another way, see later problems.]

9. For motion under a pure (rest mass preserving) inverse square law force $\mathbf{f} = -\alpha\mathbf{r}/r^3$, where α is a constant, derive the energy equation $\gamma mc^2 - \alpha/r = \text{constant}$.

Additional questions

10. Show that if a 4-vector has a component which is zero in *all* frames, then the entire vector is zero. What insight does this offer into energy and momentum?

11. A system consists of two photons, each of energy E, propagating at right angles in the laboratory frame. Find the rest mass of the system and the velocity of its CM frame relative to the laboratory frame.

Particle formation

12. A particle of mass m and energy E (in the laboratory frame) hits a free stationary target of mass M. If E is greater than a threshold energy $E_{\rm th}$, the collision produces a number of collision products with masses m_i. Show that $E_{\rm th}$ is given by

$$E_{\rm th} = \frac{(\sum_i m_i)^2 - m^2 - M^2}{2M} c^2.$$

13. A particle formation experiment creates reactions of the form $b + t \to b + t + n$ where b is an incident particle of mass m, t is a target of mass M at rest in the laboratory frame, and n is a new particle. Define the "efficiency" of the experiment as the ratio of the rest energy of the new particle to the supplied kinetic energy of the incident particle. Show that, at threshold, the efficiency thus defined is equal to

$$\frac{M}{m + M + m_n/2}.$$

tem Pion formation can be achieved by the process $p + p \to p + p + \pi^0$. A proton beam strikes a target containing stationary protons. Calculate the minimum kinetic energy which must be supplied to an incident proton to allow pions to be formed, and compare this to the rest energy of a pion.

14. A photon is incident on a stationary proton. Find, in terms of the rest masses, the threshold energy of the photon if a neutron and a pion are to emerge.

Particle decay

15. Particle tracks are recorded in a bubble chamber subject to a uniform magnetic field of 2 tesla. A vertex consisting of no incoming and two outgoing tracks is observed. The tracks lie in the plane perpendicular to the magnetic field, with radii of curvature 1.67 m and 0.417 m, and separation angle 21°. It is believed that they belong to a proton (mass 939.3 MeV/c^2) and pion (mass 139.6 MeV/c^2), respectively. Assuming this, and that the process at the vertex at the vertex is decay of a neutral particle into two products, find the rest mass of the neutral particle.

16. A particle with known rest mass M and energy E decays into two products with known rest masses m_1 and m_2. Find the energies E_1 and E_2 (in the lab frame) of the products, by the following steps:

    (i) Find the energies E_1' and E_2' of the products in the CM frame.

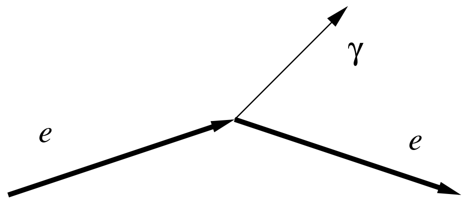
    (ii) Show that the momentum of either decay product in the CM frame is

    $$p = \frac{c}{2M}[(m_1^2 + m_2^2 - M^2)^2 - 4m_1^2 m_2^2]^{1/2}.$$

    (iii) Find the Lorentz factor and the speed v of the CM frame relative to the lab.

    (iv) Write down, in terms of v, γ, p, E_1' and E_2', expressions for E_1 and E_2 when the products are emitted (1) along the line of flight, and (2) at right angles to the line of flight in the CM frame.

17. This diagram illustrates a process in which an electron emits a photon:



    Prove that the process is impossible. Prove also that a photon cannot transform into an electron-positron pair in free space, and that a photon in free space cannot decay into a pair of photons with differing directions of propagation.

18. A decay mode of the neutral Kaon is $K^0 \to \pi^+ + \pi^-$. The Kaon has momentum 300 MeV/c in the laboratory, and one of the pions is emitted, in the laboratory, in a direction perpendicular to the velocity of the Kaon. Find the momenta of both pions. [Use 497.611 MeV/c^2 as the mass of the kaon, and 139.57061 MeV/c^2 as the mass of the pion.]

Scattering

19. Consider a head-on elastic collision between a moving "bullet" of rest mass m and a stationary target of rest mass M such that the bullet recoils in the opposite of its original direction. Show that the post-collision Lorentz factor γ of the bullet cannot exceed $(m^2 + M^2)/(2mM)$. (This means that for large energies almost all the energy of the bullet is transferred to the target, very different from the classical result.) [Hint: consider $P_t^\mu - Q_b^\mu$, where P_t^μ is the initial 4-momentum of the target and Q_b^μ is the final 4-momentum of the bullet.]

20. Particles of mass m and kinetic energy T are incident on similar particles at rest in the laboratory. Show that, if elastic scattering takes place, then the minimum angle between the final momenta in the laboratory is given by

    $$\cos\theta_{\min} = \frac{T}{T + 4mc^2}.$$

Motion under a given acceleration or force

21. Twin paradox:

    (i) Evaluate the acceleration due to gravity at the Earth's surface (9.8 m/s^2) in units of years and light-years.

    (ii) In the twin paradox, the travelling twin leaves Earth on board a spaceship undergoing motion at constant proper acceleration of 9.8 m/s^2. After 5 years of proper time for the spaceship, the direction of the rockets are reversed so that the spaceship accelerates towards Earth for 10 proper years. The rockets are then reversed again to allow the spaceship to slow and come to rest on Earth after a further 5 years of spaceship proper time. How much does the travelling twin age? How much does the stay-at-home twin age?

22. Consider a particle moving in a straight line with speed v, rapidity ρ, and proper acceleration a_0. Prove that $d\rho/d\tau = a_0/c$. [Hint: use the fact that colinear rapidities are additive.]

23. The axis of a cylinder lies along the x' axis. The cylinder has no translational motion in S', but it rotates about its axis with angular speed ω'. When observed in S the cylinder travels and rotates.

    (i) Prove that in S at any instant the cylinder is twisted, with a twist per unit length $\gamma\omega' v/c^2$. [Hint: consider the rotation of the flat surfaces at the two ends of the cylinder; a line painted on either surface rotates like the hand of a clock.]

    (ii) Is the cylinder in mechanical equilibrium? Comment on whether or not you expect there to be internal shear forces in the cylinder in frame S.

24. A "photon rocket" propels itself by emitting photons in the rearwards direction. The rocket is initially at rest with mass m. Show that when the rest mass has fallen to αm the speed (as observed in the original rest frame) is given by

    $$\frac{v}{c} = \frac{1-\alpha^2}{1+\alpha^2}.$$

    [Hint: don't bother with equations of motion, use conservation of momentum.]

    It is desired to reach a speed giving a Lorentz factor of 10. What value of α is required? Supposing the rocket cannot pick up fuel en route, what proportion of its initial mass must be devoted to fuel if it is to make a journey in which it first accelerates to $\gamma = 10$, then decelerates to rest at the destination (the destination being a star with negligible relative speed to the sun)?

25. A rocket propels itself by giving portions of its mass m a constant velocity $\mathbf{u}$ relative to its instantaneous rest frame. Let S' be the frame in which the rocket is at rest at time t. Show that, if v' is the speed of the rocket in S', then to first order in dv',

$$(-dm)u = mdv'.$$

Hence prove that, when the rocket attains a speed v relative to its initial rest frame, the ratio of final to initial rest mass of the rocket is

$$\frac{m_f}{m_i} = \left(\frac{1 - v/c}{1 + v/c}\right)^{c/2u}.$$

Note that the least expenditure of mass occurs when $u = c$, *i.e.*, the "photon rocket".

Prove that if the rocket moves with constant proper acceleration a_0 for a proper time τ, then $m_f/m_i = \exp(-a_0\tau/u)$.

26. Show that the motion of a particle in a uniform magnetic field is in general helical, with the period for a cycle independent of the initial direction of the velocity. [Hint: what can you learn from $\mathbf{f} \cdot \mathbf{v}$?]