B2: Symmetry and Relativity
Problem Set 2: Lorentz transformations
MT 2022 Week 3

Conventions: Greek indices take values 0 through 3, while Latin indices take values 1 through 3. 3-vectors can also be indicated by boldface, *e.g.*, $\mathbf{a}$. Minkowski metric $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$.

1. The 4-vector field F^μ is given by $F^\mu = 2x^\mu + k^\mu(x^\nu x_\nu)$ where k^μ is a constant 4-vector and $x^\mu = (ct, x, y, z)$ is the 4-vector displacement in spacetime. Evaluate the following:

   (i) $\partial_\lambda x^\lambda$

   (ii) $\partial^\mu(x_\lambda x^\lambda)$

   (iii) $\partial^\mu \partial_\mu x^\nu x_\nu$

   (iv) $\partial_\lambda F^\lambda$

   (v) $\partial^\mu(\partial_\lambda F^\lambda)$

   (vi) $\partial^\mu \partial_\mu \sin(k_\lambda x^\lambda)$

   (vii) $\partial^\mu x^\nu$

2. Show that the Lorentz transformations in a single spatial direction form a group.

3. Show that the matrix

$$(K_1)^\mu{}_\nu = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

   generates a boost in the x direction with $\Lambda(\eta) = e^{-i\eta K_1}$.

   Write the matrix form of the generator K_2 for infinitesimal boosts along the y axis. Multiply an infinitesimal boost along x by another along y. What does the form of the matrix indicate about whether non-aligned Lorentz transformations can form a group?

4. Show, using algebra, a spacetime diagram, or otherwise,

   (i) the temporal order of two events is the same in all reference frames if and only if they are separated by a time-like interval;

   (ii) there exists a reference frame in which two events are simultaneous if and only if they are separated by a space-like interval;

   (iii) for any time-like vector there exists a frame in which its spatial part is zero;

   (iv) any vector orthogonal to a time-like vector must be space-like;

   (v) with one exception, any vector orthogonal to a null vector is space-like, and describe the exception;

   (vi) the instantaneous 4-velocity of a particle is parallel to the worldline (*i.e.*, demonstrate that you understand the meaning of this claim—if you do then it is obvious); and

(vii) if the 4-displacement between any two events is orthogonal to an observer's worldline, then the events are simultaneous in the rest frame of that observer.

5. Define *proper time.* A worldline (not necessarily straight) may be described as a locus of time-like separated events specified by $X^\mu = (ct, x, y, z)$ in some inertial reference frame. Show that the increase of proper time τ along a given worldline is related to reference frame time t by $dt/d\tau = \gamma$.

   Two particles have 3-velocities $\mathbf{u}$ and $\mathbf{v}$ in some reference frame. The Lorentz factor for their relative 3-velocity $\mathbf{w}$ is given by

$$\gamma_w = \gamma_u \gamma_v (1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

   Prove this twice, by using each of the following two methods:

   (i) In the given frame, the worldline of the first particle is $X^\mu = (ct, \mathbf{u}t)$. Transform to the rest frame of the other particle to obtain

$$t' = \gamma_v t(1 - \mathbf{u} \cdot \mathbf{v}/c^2).$$

   Obtain dt'/dt and apply the result of the first part of this question.

   (ii) Use the invariant $U^\mu V_\mu$, first showing that it is equal to $-c^2\gamma_w$.

6. Derive a formula for the frequency ω of light waves from a moving source, in terms of the proper frequency ω_0 in the source frame and the angle in the observer's frame, θ, between the direction of observation and the velocity of the source.

   A galaxy with a negligible speed of recession from Earth has an active nucleus. It has emitted two jets of hot material with the same speed v in opposite directions, at an angle θ to the direction to the Earth. A spectral line in singly-ionised Mg (proper wavelength $\lambda_0 = 448.1$ nm) is emitted from both jets. Show that the wavelengths $\lambda_\pm$ observed on Earth from the two jets are given by

$$\lambda_\pm = \lambda_0 \gamma (1 \pm (v/c) \cos\theta)$$

   (you may assume the angle subtended at Earth by the jets is negligible). If $\lambda_+ = 420.2$ nm and $\lambda_- = 700.1$ nm, find v and θ.

   In some cases, the receding source is difficult to observe. Suggest a reason for this.

Additional questions

7. For any two future-pointing time-like vectors U^μ and V^μ, prove that $U^\mu V_\mu = -uv\cosh\rho$, where ρ is the relative rapidity of frames in which U^μ and V^μ are purely temporal and with $U^\mu U_\mu = -u^2$ and $V^\mu V_\mu = -v^2$.

8. In a given inertial frame S, two particles are shot out from a point in orthogonal directions with equal speeds v. At what rate does the distance between the particles increase in S? What is the speed of each particle relative to the other?

9. In a frame S a guillotine blade in the (x, y) plane falls in the negative y direction towards a block level with the x axis and centred at the origin. The angle of the edge of the blade is such that the point of intersection of blade and block moves at a speed in excess of c in the positive x direction. Show that in some frames S' in standard configuration with S, this point moves in the *opposite* direction along the block. [For simplicity, assume that the blade drops with constant speed u.]

   Now suppose that when the centre of the blade arrives at the block, the whole blade instantaneously evaporates in frame S (for example, it could be vapourized by a very powerful laser beam incident from the z direction). A piece of paper placed on the block is therefore cut on the negative x-axis only. Explain this in S'.

10. Derive the equations describing the transformation of velocity:

$$\begin{aligned}\mathbf{u}'_{\parallel} &= \frac{\mathbf{u}_{\parallel} - \mathbf{v}}{1 - \mathbf{u}\cdot\mathbf{v}/c^2} \\ \mathbf{u}'_{\perp} &= \frac{\mathbf{u}_{\perp}}{\gamma_v(1 - \mathbf{u}\cdot\mathbf{v}/c^2)}\end{aligned}$$