THIRD YEAR PHYSICS COLLECTIONS

HILARY TERM

B2: SYMMETRY AND RELATIVITY

SOLUTION NOTES

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1. Write down the Lorentz transformation appropriate for a 4-vector involving the time t and position x, y, z of an event, for a pair of inertial reference frames with aligned axes in relative motion along the x direction, and give an expression relating the quantities γ and β appearing in the transformation. By using the metric, or otherwise, show that for any pair of 4-vectors A^μ, B^μ, the quantity $A^\mu B_\mu$ is Lorentz invariant. Show that any 4-vector orthogonal to a time-like 4-vector must be space-like. [9]

(i) Using $U^\mu V_\mu$, or otherwise, establish that

$$\gamma(w) = \gamma(u)\gamma(v)(1 - \mathbf{u}\cdot\mathbf{v}/c^2)$$

where $\mathbf{u}$ and $\mathbf{v}$ are velocities of a pair of particles in a given frame, and $\mathbf{w}$ is their relative velocity. [3]

(ii) Prove that the 4-acceleration and 4-velocity of a particle are always orthogonal, *i.e.*, $U^\mu A_\mu = 0$. [4]

(iii) Show that

$$\frac{d}{dv}(\gamma v) = \gamma^3$$

and that

$$a_0^2 = \gamma^4 a^2 + \gamma^6(\mathbf{v}\cdot\mathbf{a})^2/c^2$$

where $\mathbf{a}$ is the 3-acceleration of a particle moving at velocity $\mathbf{v}$, and $\mathbf{a}_0$ is its proper 3-acceleration. Hence, or otherwise, show that a particle subject to a constant force $\mathbf{f}$ in the laboratory frame, and moving in the direction of the force, has a constant proper acceleration. [9]

2. The 4-momentum (energy-momentum 4-vector) of a single particle is defined by $P^\mu \equiv mU^\mu$ where $U^\mu \equiv dX^\mu/d\tau$. Define the symbols m, τ, X^μ appearing in this expression, and prove that the components P^μ can be written $P^\mu = (\gamma mc, \gamma m\mathbf{v})$ where $\mathbf{v}$ is the velocity of the particle and γ is the Lorentz factor. Now consider two particles A and B with energies E_A and E_B and momenta $\mathbf{p}_A$ and $\mathbf{p}_B$, respectively. What can you say about the quantity $E_A E_B - \mathbf{p}_A \cdot \mathbf{p}_B c^2$? [7]

(a) Particle Y of mass m_Y decays at rest into particles A and C with masses m_A and m_C, respectively. Derive an expression for the energy of particle C in the lab frame in terms of the particle masses. [4]

(b) Now consider a three-body decay of particle Y at rest into products A, B (of mass m_B), and C, all of which have non-zero rest mass. By considering A and B as a composite particle X, or otherwise, show that the energy of C in the lab frame is

$$E_C = \frac{(m_Y^2 + m_C^2 - m_A^2 - m_B^2)c^4 - 2E_A E_B + 2\mathbf{p}_A \cdot \mathbf{p}_B c^2}{2m_Y c^2}.$$

Give an expression for the maximum energy of particle C. [5]

(c) By taking the limit as $m_B \to 0$, find an expression for the maximum energy of particle C when particle B is massless. [1]

(d) The trouble with the way the answer to part (c) was obtained is that massless particles may only travel at the speed of light. Comment on why the answer to part (c) is nevertheless correct. [3]

(e) The momenta of charged particles are sometimes measured by observing their tracks in a region of known uniform magnetic field. Write down the equation of motion and find the relationship between the momentum and the radius of curvature of the track, for a particle whose initial velocity is perpendicular to the magnetic field. [5]

3. Frame S' moves with a constant 3-velocity $\mathbf{v} = (v_x, 0, 0)$ relative to the lab frame S. In S, the components of the electric field and the magnetic field are, respectively, $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$. Find the electric (E'_x, E'_y, E'_z) and magnetic (B'_x, B'_y, B'_z) field components in the frame S'. [3]

Define a 4-vector potential A^μ and a 4-wave vector K^μ. A plane, linearly polarised electromagnetic wave propagates in the z direction through vacuum. Using the 4-vector potential $A^\mu = (0, 0, A_y, 0)$, where

$$A_y = -A_0 \left(\frac{\sin(K^\nu X_\nu)}{\omega/c} \right),$$

with A_0 a constant and ω the frequency, find the field strength tensor

$$F^{\alpha\beta} = \partial^\alpha A^\beta \partial^\beta A^\alpha.$$

Define the 4-current J^μ. Write down the 4-current continuity condition in 3-vector form and 4-vector form. Using Maxwell's equations written in 3-vector form, show that $\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$. The Lorentz force acting on a unit volume of charge density ρ can be written as $f^\mu = F^{\mu\nu} J_\nu$. What is the physical meaning of the f^0 component of this 4-vector? [7]

A 4-current $J^\mu = (\rho c, j_x, 0, 0)$, where $j_x = \rho v_0$, flows along an infinitely long straight wire which is stationary in the lab frame S. Find the electric and magnetic fields generated in the lab frame S, in the frame moving perpendicular to the wire with velocity $\mathbf{u} = (0, v_0, 0)$, and in the frame co-moving with the electrons, *i.e.*, having 3-velocity $\mathbf{u} = (v_0, 0, 0)$ in S. [7]

During the head-on collision between a low-energy photon defined by 4-wave vector $K^\mu = (\omega/c, k_x, 0, 0)$ and an ultra-relativistic particle of rest mass M_0 and total energy W, an inverse Compton scattering was observed. Show that the maximum energy, which the photon can gain during the process, can be estimated as

$$E_{ph}^{max} = 4 \left(\frac{W}{M_0 c^2} \right)^2 E_{ph}$$

where E_{ph} is the initial energy of the photon. What condition defines "low energy" for the photon? [8]

4. The 4-force is defined to be

$$F^{\mu} = \left(\frac{\gamma}{c} \frac{dE}{dt}, \gamma \mathbf{f} \right)$$

where γ is the Lorentz factor, E is the energy, and the 3-force $\mathbf{f} = d\mathbf{p}/dt$ where $\mathbf{p}$ is the 3-momentum. If U^{μ} is the 4-velocity, find the conditions under which $U^{\mu} F_{\mu} = 0$ and show that this leads to the classical relation between force and the rate of doing work. [6]

Consider a 4-force F^{μ} applied to a particle travelling with 3-velocity $\mathbf{u}$ in reference frame S. Now consider a reference frame, S', travelling with 3-velocity $\mathbf{v}$ relative to S. Show that in reference frame S', for the case of a pure force (where $dm_0/dt = 0$), the component of the force parallel to the relative velocity of the reference frames, $\mathbf{f}_{\parallel}$, transforms to

$$\mathbf{f}'_{\parallel} = \frac{\mathbf{f}_{\parallel} - \mathbf{v}(\mathbf{f} \cdot \mathbf{u})/c^2}{1 - \mathbf{u} \cdot \mathbf{v}/c^2}.$$

[12]

For a pure force, show that the 3-force is not necessarily parallel to the 3-acceleration. Show that, in fact,

$$\mathbf{f} = \gamma m_0 \mathbf{a} + \frac{\mathbf{f} \cdot \mathbf{u}}{c^2} \mathbf{u}.$$

[7]

1. For a Lorentz transf. - a boost along OX - we have $A^\mu \to A'^\mu$:

$$A'^0 = \gamma\,(A^0 - \beta A^1)$$
$$A'^1 = \gamma\,(A^1 - \beta A^0)$$
$$A'^2 = A^2$$
$$A'^3 = A^3$$

Here $\gamma = 1/\sqrt{1-\beta^2}$, $\beta = |\bar{V}|/c$, $\bar{V}$ is the velocity of S' w.r.t. S.

- $A^\mu B_\mu = \eta_{\mu\nu} A^\mu B^\nu = A'^\mu B'_\mu$?

Yes: $A'^\mu = \dfrac{\partial x'^\mu}{\partial x^\rho} A^\rho \qquad B'_\mu = \dfrac{\partial x^\varkappa}{\partial x'^\mu} B_\varkappa$

$$A'^\mu B'_\mu = \underbrace{\frac{\partial x'^\mu}{\partial x^\rho}\frac{\partial x^\varkappa}{\partial x'^\mu}}_{\delta_\rho^\varkappa} A^\rho B_\varkappa = A^\rho B_\rho = A^\mu B_\mu .$$

Note: this is true for any metric $g_{\mu\nu}$, not just $\eta_{\mu\nu}$, since $g_{\mu\nu} A^\mu B^\nu$ is a scalar. For Lor. transf. $\dfrac{\partial x'^\mu}{\partial x^\rho} = \Lambda^\mu{}_\rho = \text{const}$

(2)

Specifically, $\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

$A' = \Lambda A \qquad B' = \Lambda B$

In components, $A'^{\mu} = \Lambda^{\mu}{}_{\rho} A^{\rho}$ and $B'^{\nu} = \Lambda^{\nu}{}_{\varkappa} B^{\varkappa}$

The inverse transf. is given by

$$A = \Lambda^{-1} A', \qquad B = \Lambda^{-1} B',$$

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that $\Lambda^{T} = \Lambda$

Then $A^{\mu} B_{\mu} = \eta_{\mu\nu} A^{\mu} B^{\nu} =$

$= \eta_{\mu\nu} \Lambda^{-1\,\mu}{}_{\lambda} A'^{\lambda} \Lambda^{-1\,\nu}{}_{\varkappa} B'^{\varkappa}$

But $\eta_{\mu\nu} = \Lambda^{\rho}{}_{\mu} \Lambda^{\sigma}{}_{\nu} \eta'_{\rho\sigma} \Rightarrow$

$A^{\mu} B_{\mu} = \Lambda^{\rho}{}_{\mu} \Lambda^{-1\,\mu}{}_{\lambda} \Lambda^{\sigma}{}_{\nu} \Lambda^{-1\,\nu}{}_{\varkappa} A'^{\lambda} B'^{\varkappa} \eta'_{\rho\sigma} =$

$$= \eta'_{\rho\sigma} A'^{\rho} B'^{\sigma} = A'^{\mu} B'_{\mu}.$$

- Any 4-vector orthogonal to a time-like 4-vector must be space-like.

If B is a time-like vector $\Rightarrow B^2 < 0$

$\Rightarrow -(B^0)^2 + \bar{B}^2 < 0 \Rightarrow |B^0| > |\bar{B}|$

If A is such that $A \cdot B = 0 \Rightarrow$

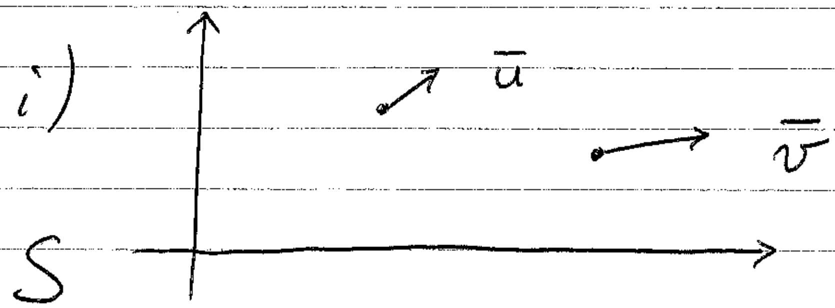
$-A^0 B^0 + \vec{A} \cdot \vec{B} = 0 \Rightarrow |A^0||B^0| = |\bar{A}||\bar{B}||\cos\varphi|$

then $\dfrac{|A^0|}{|\bar{A}|} = \dfrac{|\bar{B}|}{|B^0|} |\cos\varphi| < 1$ (since

$|\cos\varphi| \leq 1$ and $\dfrac{|\bar{B}|}{|B^0|} < 1$) $\Rightarrow$

$A^2 > 0$ (space-like)

i)



In S: $U^{\mu} = (\gamma(u) c, \gamma(u) \bar{u})$

$\Rightarrow$

$V^{\mu} = (\gamma(v) c, \gamma(v) \bar{v})$

$$U^\mu V_\mu = -c^2 \gamma(u)\gamma(v) + \gamma(u)\gamma(v)\,\bar{u}\cdot\bar{v}$$

In the frame S' – associated e.g. with the particle moving with $\bar{u}$ (in S):

$$U'^\mu = (c, \bar{0})$$

$$V'^\mu = (\gamma(w)c, \gamma(w)\,\bar{w}),$$

where $\bar{w}$ is the 3-velocity of the particle „$\bar{v}$" in S'. Since $U\cdot V = U'V'$, we get

$$U'\cdot V' = -\gamma(w)c^2 = U\cdot V = -c^2\gamma(u)\gamma(v)\times$$
$$\times\left(1 - \frac{\bar{u}\cdot\bar{v}}{c^2}\right)$$

$$\Rightarrow \boxed{\gamma(w) = \gamma(u)\gamma(v)\left(1 - \frac{\bar{u}\cdot\bar{v}}{c^2}\right)}.$$

ii) Since $U^\mu U_\mu = \eta_{\mu\nu}\dfrac{dx^\mu}{d\tau}\dfrac{dx^\nu}{d\tau} =$

$$= \frac{-c^2dt^2 + d\bar{x}^2}{d\tau^2} = \frac{ds^2}{d\tau^2} = \frac{-c^2d\tau^2}{d\tau^2} = -c^2,$$

taking a derivative w.r.t. τ of $U^\mu U_\mu$ gives

$$0 = \frac{d}{d\tau}\left(U^\mu U_\mu\right) = 2U^\mu \frac{dU_\mu}{d\tau} = 2U \cdot A$$

$\Rightarrow$ $U \cdot A = 0$, where $A^\mu = dU^\mu/d\tau$ is the 4-acceleration.

iii) $$\frac{d}{dv}(\gamma v) = \gamma + v\frac{d\gamma}{dv}$$

$$\frac{d\gamma}{dv} = \frac{v}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} \Rightarrow$$

$$\frac{d}{dv}(\gamma v) = \gamma + \frac{v^2}{c^2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2} =$$

$$= \left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(\frac{v^2}{c^2} + 1 - \frac{v^2}{c^2}\right) = \gamma^3.$$

Thus, $\frac{d}{dv}(\gamma v) = \gamma^3$.

To prove the other identity, recall that $U^\mu = (\gamma(v)c, \gamma(v)\bar{v})$ in S, and $A^\mu = dU^\mu / d\tau$, $d\tau = dt/\gamma$,

so $A^\mu = \left(c \dfrac{d\gamma}{d\tau}, \dfrac{d}{d\tau}(\gamma \bar{v}) \right) =$

$= \left(c\gamma\dot{\gamma}, \gamma(\dot{\gamma\bar{v}}) \right)$, where $\dot{\gamma} = d\gamma/dt$.

But $\dot{\gamma} = \dfrac{d}{dt}\left[\left(1 - \dfrac{\bar{v}^2}{c^2}\right)^{-1/2} \right] = \dfrac{v^i}{c^2}\left(1 - \dfrac{\bar{v}^2}{c^2}\right)^{-3/2} \dot{v}^i =$

$= \dfrac{\bar{v}\cdot\bar{a}}{c^2}\gamma^3$.

Thus, $A^\mu = \left(\gamma^4 \dfrac{\bar{v}\cdot\bar{a}}{c}, \gamma^4 \dfrac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2 \bar{a} \right)$.

The proper acceleration (the one in the particles „own" frame) is

$$A_0^\mu = (0, \bar{a}_0) \qquad (\gamma = 1, \bar{v} = 0)$$

Consider $A_0 \cdot A_0 = A \cdot A$;

$$a_0^2 = -\gamma^8 \frac{(\bar{v}\cdot\bar{a})^2}{c^2} + \left(\gamma^4 \frac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2 \bar{a} \right)^2 =$$

$$= -\gamma^8 \frac{(\bar{v}\cdot\bar{a})^2}{c^2} + \gamma^8 \frac{(\bar{v}\cdot\bar{a})^2}{c^4}\bar{v}^2 + 2\gamma^6 \frac{(\bar{v}\cdot\bar{a})^2}{c^2} + \gamma^4 \bar{a}^2.$$

But $\gamma^2 \frac{v^2}{c^2} = \gamma^2 - 1 \Rightarrow \frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$

$$\Rightarrow \boxed{\bar{a}_0^{\,2} = \gamma^6 \frac{(\bar{v}\cdot\bar{a})^2}{c^2} + \gamma^4 \bar{a}^2}$$

Finally, if $\bar{v}$ and $\bar{a}$ have the same direction:

$$a_0^2 = a^2\gamma^4 + \gamma^6 \frac{v^2 a^2}{c^2} = a^2\gamma^4 + a^2\gamma^6\left(1 - \frac{1}{\gamma^2}\right) =$$

$$= a^2\gamma^6 \qquad \left(a_0 \equiv |\vec{a}_0|,\ a \equiv |\vec{a}|\right)$$

We need to show that $a^2\gamma^6 = \text{const}$ for a particle subject to a constant force and moving along $\bar{a}$. Indeed,

$$m\frac{d}{dt}(\gamma\bar{v}) = \bar{f} = \text{const}$$

implies $\dot{\gamma}\bar{v} + \gamma\bar{a} = \bar{f}/m = \text{const}$ or

$$\dot{\gamma}^2 v^2 + 2\dot{\gamma}\gamma\,\bar{v}\cdot\bar{a} + \gamma^2 a^2 = \text{const}$$

Recall that $\dot{\gamma} = \frac{\bar{v}\cdot\bar{a}}{c^2}\gamma^3 \Rightarrow$

$$\frac{(\bar{v}\cdot\bar{a})^2}{c^4}\gamma^6 v^2 + 2\gamma\frac{(\bar{v}\cdot\bar{a})^2}{c^2}\gamma^3 + \gamma^2 a^2 = \text{const}$$

For $\bar{v}$ along $\bar{a}$ this means

$$\left(\frac{v^2}{c^2}\right)^2 a^2\gamma^6 + 2\gamma^4\frac{v^2}{c^2}a^2 + \gamma^2 a^2 = \text{const}$$

Using $\gamma^2\frac{v^2}{c^2} = \gamma^2 - 1$ again, this can be reduced to

$$a^2\gamma^2(\gamma^2-1)^2 + 2\gamma^4 a^2 - \gamma^2 a^2 = \text{const}$$

$$\Rightarrow a^2\gamma^6 = \text{const}$$

Thus, $a_0^2 = a^2\gamma^6 = \text{const} \Rightarrow a_0 = \text{const}$.

2. $P^\mu = m U^\mu = m \dfrac{dX^\mu}{d\tau}$

– the 4-momentum (in some frame S) of a particle with mass m moving along the trajectory specified by $X^\mu(t) = (ct, x(t), y(t), z(t))$. Here τ is the proper time of the particle defined by

$$-c^2 d\tau^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \quad d\tau = \frac{dt}{\gamma}.$$

$$p^\mu = m\frac{dX^\mu}{d\tau} = m\gamma \frac{dX^\mu}{dt} = (mc\gamma, \gamma m\vec{v}),$$

where $v^i = \dot{x}^i$, $i = 1, 2, 3$.

$p^\mu = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$, where $\mathcal{E} = \gamma mc^2$, $\vec{p} = \gamma m\vec{v}$.

For $P^\mu_A = \left(\frac{\mathcal{E}_A}{c}, \vec{p}_A\right)$, $P^\mu_B = \left(\frac{\mathcal{E}_B}{c}, \vec{p}_B\right)$

$P_A \cdot P_B = -\dfrac{\mathcal{E}_A \mathcal{E}_B}{c^2} + \vec{p}_A \cdot \vec{p}_B$ is Lorentz-invar. as a scalar product of 2 4-vectors

$\Rightarrow \mathcal{E}_A \mathcal{E}_B - \vec{p}_A \vec{p}_B c^2 = -c^2 P_A P_B$ is Lor. invar.

a) $Y \to A + C$

4-momentum conservation implies

$$p_Y = p_A + p_C$$

$$\Rightarrow \quad p_A = p_Y - p_C \Rightarrow p_A^2 = (p_Y - p_C)^2$$

Since $p^2 = -m^2c^2$, we have

$$-m_A^2 c^2 = -m_Y^2 c^2 - m_c^2 c^2 - 2 p_Y p_c$$

In the lab frame, $p_c = (\mathcal{E}_c/c, \vec{p}_c)$, $p_Y = (m_Y c, \vec{0})$,

so $p_Y p_c = -\mathcal{E}_c \cdot m_Y \Rightarrow$

$$m_A^2 = m_Y^2 + m_c^2 - \frac{2\mathcal{E}_c m_Y}{c^2}$$

$$\Rightarrow \boxed{\mathcal{E}_c = \frac{(m_Y^2 + m_c^2 - m_A^2)}{2 m_Y} c^2}$$

b) $Y \to A + B + C$

Using the same approach as in a) we find

$$p_Y = p_A + p_B + p_C$$

$$(p_Y - p_C)^2 = p_A^2 + p_B^2 + 2 p_A p_B$$

$$-m_Y^2c^2 - m_c^2c^2 = -m_A^2c^2 - m_B^2c^2 + 2p_Ap_B - 2p_Yp_c$$

$$\Rightarrow \mathcal{E}_c = \frac{(m_Y^2 + m_c^2 - m_A^2 - m_B^2)c^2 + 2p_Ap_B}{2m_Y}$$

This expression has a maximum at max value of p_Ap_B (which is Lor.-invar. and can be evaluated in any inertial ref. frame, e.g. in the rest frame of particle B, where it is equal $-\mathcal{E}_A' m_B$). The max of $p_Ap_B = -\mathcal{E}_A' m_B$ is at min of $\mathcal{E}_A'$, i.e. at $\mathcal{E}_A' = m_Ac^2$, thus $(p_Ap_B)_{max} = -m_Am_Bc^2$. The max energy $\mathcal{E}_c^{max}$ is

$$\boxed{\mathcal{E}_c^{max} = \frac{m_Y^2 + m_c^2 - (m_A + m_B)^2}{2m_Y}c^2}$$

c) $$\mathcal{E}_c^{max} \longrightarrow \frac{m_Y^2 + m_c^2 - m_A^2}{2m_Y}c^2 \quad \text{for } m_B \to 0$$

d) if $m_B = 0$, we can repeat the argument by computing $p_A p_B$ in particle A's rest frame (getting $p_A p_B = -\mathcal{E}_B' m_A$ with max value being zero). If $m_A = 0$ as well, $p_A p_B = \frac{\mathcal{E}_A \mathcal{E}_B}{c^2}(\cos\varphi - 1)$, where φ is the angle between $\vec{p}_A$ and $\vec{p}_B$.

The max value of that last expression is zero. So in any case $(p_A p_B)_{max} = 0$ for $m_B = 0$ and the result in c) remains valid.

The equation of motion is

$$\frac{d\vec{p}}{dt} = \vec{f} = q\,\vec{v} \times \vec{B}$$

Recall that $\frac{dP^\mu}{d\tau} = F^\mu = \left(\frac{\gamma \dot{\mathcal{E}}}{c}, \gamma \vec{f}\right)$

$$U^\mu = (\gamma c, \gamma \vec{v}) \Rightarrow U^\mu F_\mu = -\gamma^2 \dot{\mathcal{E}} + \gamma^2 \vec{f} \cdot \vec{v} =$$

$$= -\dot{m} c^2 = 0$$

$\Rightarrow \dot{\mathcal{E}} = \vec{f} \cdot \vec{v}$. For $\vec{f}$ given by $\vec{f} =$

$= q\,\vec{v} \times \vec{B}$, $\quad \vec{f} \cdot \vec{v} = 0 \quad \Rightarrow \dot{\mathcal{E}} = 0 \Rightarrow$

$\Rightarrow \dot{\gamma} = 0 \quad \Rightarrow |\vec{v}| = \text{const}$.

So the e.o.m. becomes $\dfrac{d\vec{p}}{dt} = \dfrac{d(\gamma m \vec{v})}{dt} =$

$= \gamma m \dot{\vec{v}} = q\,\vec{v} \times \vec{B}$

$\Rightarrow \dot{\vec{v}} = \dfrac{q}{\gamma m}\,\vec{v} \times \vec{B} \quad \Rightarrow \quad \vec{a} \cdot \vec{v} = 0$

$(\vec{a} \equiv \dot{\vec{v}})$.

It's a motion along a circle $(\vec{a} \perp \vec{v})$, $\vec{v} \perp \vec{B}$, with constant speed $v = |\vec{v}|$ and acceleration $\dfrac{v^2}{R} = \dfrac{q v B}{\gamma m}$

$\Rightarrow R = \dfrac{\gamma m v}{qB} = \dfrac{p_\perp}{qB}$

3. S' moves with velocity $\bar{v} = (v_x, 0, 0)$ relative to S.

Transformation laws for $\bar{E}, \bar{B}$ are

$$\bar{E}'_{\parallel} = \bar{E}_{\parallel}$$

$$\bar{E}'_{\perp} = \gamma(E_{\perp} + \bar{v} \times \bar{B})$$

$$\bar{B}'_{\parallel} = B_{\parallel}$$

$$\bar{B}'_{\perp} = \gamma(\bar{B}_{\perp} - \bar{v} \times \bar{E}/c^2)$$

In coordinates of S:

$$E'_x = E_x$$

$$E'_y = \gamma(E_y - v_x B_z)$$

$$E'_z = \gamma(E_z + v_x B_y)$$

$$B'_x = B_x$$

$$B'_y = \gamma(B_y + v_x E_z/c^2)$$

$$B'_z = \gamma(B_z - v_x E_y/c^2)$$

$$A^{\mu} = (\phi/c, \bar{A}), \text{ where}$$

$$\bar{B} = \text{curl } \bar{A} = \nabla \times \bar{A}$$

$$\bar{E} = -\nabla\phi - \partial\bar{A}/\partial t$$

$$K^{\mu} = \left(\frac{\omega}{c}, \bar{K}\right) \qquad K^{\mu}K_{\mu} = 0 \text{ for photons.}$$

$$\Rightarrow |\bar{K}| = \omega/c$$

For the electromagnetic wave with

$$A^{\mu} = (0, 0, A_y, 0), \text{ where}$$

$$A_y = -A_0 \frac{\sin K^{\mu}X_{\mu}}{\omega/c}$$

one can find $F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$ as follows: $F^{02} = -F^{20}$, $F^{12} = -F^{21}$, $F^{32} = -F^{23}$ are the only non-zero components, given A^{μ} with only $A_y \neq 0$.

$$A_y = \frac{A_0 c}{\omega} \sin(\omega t - kz)$$

$$F^{02} = \partial^0 A^y = \frac{\partial}{\partial x_0} A^y = -\frac{1}{c}\partial_t A^y =$$

$$= -A_0 \cos(\omega t - kz) = +E_y/c.$$

$$F^{12} = \partial^x A^y = 0$$

$$F^{32} = \partial^z A^y = \frac{\partial}{\partial z} A^y = -A_0 \cos(\omega t - kz)$$

$$= -B_x.$$

Now, the 4-current is $J^\mu = (\rho c, \bar{j})$, where ρ is the charge density and $\vec{j}$ is the current density.

<u>Continuity eq</u>: $\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \operatorname{div} \bar{j} = 0.$

Maxwell eqs:

$$\operatorname{div} \bar{E} = \rho/\varepsilon_0$$

$$\operatorname{curl} \bar{B} = \mu_0 \bar{J} + \varepsilon_0 \mu_0 \frac{\partial \bar{E}}{\partial t}$$

The covariant form is $\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$

$\alpha = 0: \quad \partial_x F^{01} + \partial_y F^{02} + \partial_z F^{03} = \mu_0 \rho c$

$F^{0i} = E_i / c \quad \Rightarrow \quad \operatorname{div} \bar{E} = \mu_0 c^2 \rho$

With $c^2 = 1/\mu_0 \varepsilon_0 \quad \Rightarrow \quad \operatorname{div} \bar{E} = \rho / \varepsilon_0$

$\alpha = 1: \quad \partial_0 F^{10} + \partial_y F^{12} + \partial_z F^{13} = \mu_0 J^x$

$$-\frac{1}{c^2} \partial_t E_x + \partial_y B_z - \partial_z B_y = \mu_0 J^x$$

which is exactly the x-component of

$$\operatorname{curl} \bar{B} = \mu_0 \bar{J} + \varepsilon_0 \mu_0 \, \partial \bar{E} / \partial t$$

Components $\alpha = 2, 3$ are similar.

<u>Note</u>: $F^{0i} = E^i / c$

$$\left.\begin{array}{l} F^{12} = B_z \\ F^{23} = B_x \\ F^{13} = -B_y \end{array}\right\} \quad F^{ij} = \varepsilon_{ijk} B_k$$

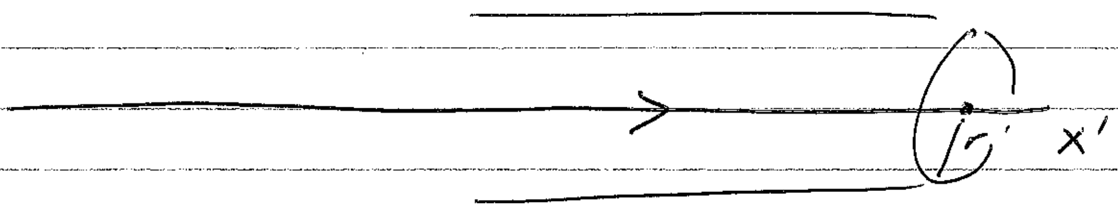
$f^\mu = F^{\mu\nu} J_\nu \; ; \; f^0 = F^{0i} J_i = \vec{E} \cdot \vec{J} / c$

(power density)

$J^\mu = (\rho c, j_x, 0, 0)$ $\quad j_x = \rho v_0$

in S.

In the comoving frame S', the wire is stationary $\Rightarrow$ source of electric field $\bar{E}'$ whose components can be found via Gauss' theorem



$$2\pi r l E_r = l\rho'/\varepsilon_0$$

$$\Rightarrow E_r' = \frac{\rho'}{2\pi\varepsilon_0 r'}$$

$$\begin{cases} E_x' = 0 \\ E_y' = \dfrac{\partial y'}{\partial r'} E_r' = E_r' \dfrac{y'}{r} \\ E_z' = \dfrac{\partial z'}{\partial r'} E_r' = E_r' \dfrac{z'}{r} \end{cases}$$

$$\Rightarrow E_y' = \frac{\rho' y'}{2\pi\varepsilon_0 r'^2}$$

$$E_z' = \frac{\rho' z'}{2\pi\varepsilon_0 r'^2} \qquad r'^2 = y'^2 + z'^2.$$

To find ρ': $S \to S'$ implies

$$J'^0 = \gamma(J^0 - \beta J^1), \text{ where}$$

$J^0 = \rho c, \quad J'^0 = \rho' c, \quad J' = j_x = \rho v_0,$

$\beta = v_0/c \Rightarrow \rho' = \rho/\gamma, \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}.$

also, $y' = y, \quad z' = z$:

$$\begin{cases} E'_x = 0 \\ E'_y = \dfrac{\rho y}{\gamma 2\pi\varepsilon_0 (y^2+z^2)} \\ E'_z = \dfrac{\rho z}{\gamma 2\pi\varepsilon_0 (y^2+z^2)} \\ \bar{B}' = 0 \end{cases}$$

To find the fields in the lab frame S, make a Lor transf. $S' \to S$:

$$E_x = 0$$

$$E_y = \gamma E'_y = \frac{\rho y}{2\pi\varepsilon_0 (y^2+z^2)}$$

$$E_z = \gamma E'_z = \frac{\rho z}{2\pi\varepsilon_0 (y^2+z^2)}$$

$$B_x = 0$$

$$B_y = -\frac{\gamma v_0 E_z'}{c^2} = -\frac{v_0 \rho z}{2\pi\varepsilon_0 c^2 (y^2+z^2)}$$

$$B_z = +\gamma \frac{v_0}{c^2} E_y' = \frac{v_0 \rho y}{2\pi\varepsilon_0 c^2 (y^2+z^2)}$$

Now one can find fields in the system S'' moving with $\bar{v} = (0, v_0, 0)$ w.r.t. S:

$$E_x'' = \gamma (E_x + v_y B_z) = -\frac{\gamma v_0^2 \rho y}{2\pi\varepsilon_0 c^2 (y^2+z^2)}$$

$$E_y'' = E_y = \frac{\rho y}{2\pi\varepsilon_0 (y^2+z^2)}$$

$$E_z'' = \gamma (E_z - v_y B_x) = \frac{\gamma \rho z}{2\pi\varepsilon_0 (y^2+z^2)}$$

$$B_x'' = \gamma (B_x - v_y E_z / c^2) = -\frac{\gamma v_0 \rho z}{2\pi\varepsilon_0 c^2 (y^2+z^2)}$$

$$B_y'' = B_y = -\frac{v_0 \rho z}{2\pi\varepsilon_0 c^2 (y^2+z^2)}$$

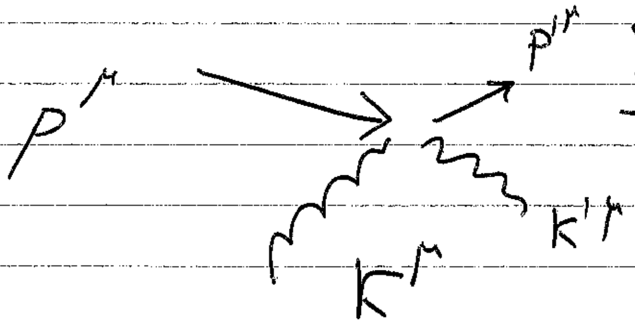
$$B_z'' = \gamma (B_z + v_y E_x / c^2) = \frac{\gamma v_0 \rho y}{2\pi\varepsilon_0 c^2 (y^2+z^2)}$$

Note that $c^2 = 1/\varepsilon_0 \mu_0$.

Alternatively, one may transform $F^{\mu\nu}$ from S' to S and S'' in the usual way, e.g. $F = \Lambda^T F' \Lambda$,

where $$\Lambda = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \text{ etc.}$$

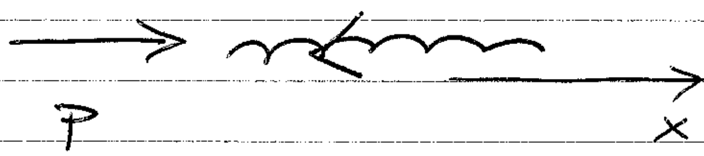
$$K^\mu = (\omega/c, K_x, 0, 0)$$



Before: $p^\mu = \left(\frac{W}{c}, p_x, 0, 0\right)$

$K^\mu = \left(\frac{\varepsilon}{c}, -\frac{\varepsilon}{c}, 0, 0\right)$

Head-on collision:



Max energy (momentum) a photon can gain $K'^\mu = \left(\frac{\varepsilon'}{c}, \frac{\varepsilon'}{c}, 0, 0\right)$

$$p + K = p' + K'$$

We are interested in k', not p' $\Rightarrow$

$$p + k - k' = p'$$

$$(p + k - k')^2 = p'^2 = -M_0^2 c^2$$

$$\cancel{p^2} + 2p(k - k') + \underbrace{\cancel{k^2}}_{=0} - 2kk' + \underbrace{\cancel{k'^2}}_{=0} = \cancel{-M_0^2 c^2}$$

$$pk - pk' - kk' = 0$$

$$-\frac{W\mathcal{E}}{c^2} - \frac{p_x \mathcal{E}}{c} + \frac{W\mathcal{E}'}{c^2} - \frac{p_x \mathcal{E}'}{c} + \frac{\mathcal{E}\mathcal{E}'}{c^2} + \frac{\mathcal{E}\mathcal{E}'}{c^2} = 0$$

$$-W\mathcal{E} - p_x c\,\mathcal{E} = -W\mathcal{E}' + p_x c\,\mathcal{E}' - 2\mathcal{E}\mathcal{E}'$$

$$(W + p_x c)\mathcal{E} = \mathcal{E}'(W - p_x c + 2\mathcal{E})$$

$$\mathcal{E}' = \frac{W + p_x c}{W - p_x c + 2\mathcal{E}}\,\mathcal{E}$$

Since $W^2 = p_x^2 c^2 + M_0^2 c^4 \Rightarrow$

$$p_x c = W\left(1 - M_0^2 c^4 / 2W^2 + \dots\right)$$

$$\Rightarrow \mathcal{E}' = \frac{2W - \dfrac{M_0^2 c^4 W}{2W^2}}{2\mathcal{E} + M_0^2 c^4 W / 2W^2}\,\mathcal{E} \Rightarrow$$

$$\mathcal{E}' = \frac{2W - M_0^2c^4/2W}{2\mathcal{E} + M_0^2c^4/2W}\,\mathcal{E} \approx$$

$$\approx 4W^2/M_0^2c^4 \cdot \mathcal{E}$$

$\Rightarrow \mathcal{E}' = \dfrac{4W^2}{M_0^2c^4}\,\mathcal{E}$ for low-energy

photons with $\mathcal{E} \ll M_0^2c^4/4W$.

4. $F^\mu = \left(\frac{\gamma}{c} \frac{d\mathcal{E}}{dt}, \gamma \bar{f} \right)$

$\frac{dp^\mu}{d\tau} = F^\mu \qquad p^\mu = m u^\mu$

$U^\mu = (\gamma c, \gamma \bar{v})$ and $\bar{f} = d\bar{p}/dt$.

We have $U \cdot F = -\gamma^2 \dot{\mathcal{E}} + \gamma^2 \bar{f} \cdot \bar{v}$

Since $U \cdot F$ is Lor-invar, we can compute it in particle's rest frame, where

$U^\mu = (c, \bar{0})$ and $F^\mu = (\dot{m} c, \bar{f}_0)$:

$U \cdot F = -\dot{m} c^2 = 0$ for constant m.

$\Rightarrow \dot{\mathcal{E}} = \bar{f} \cdot \bar{v}$.

Now, in S: $f_x = dp_x/dt$,

$f_y = dp_y/dt$, $f_z = dp_z/dt$

In S': $f_x' = dp_x'/dt'$ etc., where

$p_x' = \gamma(v)(p_x - \beta_v p_0) \Rightarrow$

$$\Rightarrow \frac{dp'_x}{dt'} = \gamma(v)\left(\frac{dp_x}{dt}\frac{dt}{dt'} - \frac{\beta_v}{c}\frac{d\mathcal{E}}{dt}\frac{dt}{dt'}\right) =$$

$$= \gamma(v)\left(f_x - \frac{v}{c^2}\dot{\mathcal{E}}\right)\frac{dt}{dt'}$$

Also, $ct' = \gamma(v)\left(ct - \beta_v x\right)$

$$\Rightarrow dt' = \gamma(v)\left(dt - \frac{\beta_v}{c}\frac{dx}{dt}dt\right) =$$

$$= \gamma(v)\left(1 - \frac{v}{c^2}u_x\right)dt$$

$$\Rightarrow dt' = dt\,\gamma(v)\left(1 - \frac{\bar{v}\cdot\bar{u}}{c^2}\right)$$

$$\Rightarrow f'_x = \frac{f_x - \frac{v}{c^2}\dot{\mathcal{E}}}{1 - \frac{\bar{v}\cdot\bar{u}}{c^2}}, \qquad \dot{\mathcal{E}} = \vec{f}\cdot\vec{u}$$

$$f'_x = \frac{f_x - \frac{v}{c^2}(\bar{f}\cdot\bar{u})}{1 - \frac{\bar{v}\cdot\bar{u}}{c^2}} \qquad \text{or}$$

$$\boxed{\bar{f}'_{\parallel} = \frac{\bar{f}_{\parallel} - \bar{v}\,(\bar{f}\cdot\bar{u})/c^2}{1 - \bar{v}\cdot\bar{u}/c^2}}$$

also, $f_y' = \frac{dp_y'}{dt'} = \frac{dp_y}{dt}\frac{dt}{dt'} \Rightarrow$

$$f_y' = \frac{f_y}{\gamma(v)\left(1 - \frac{\bar{v}\cdot\bar{u}}{c^2}\right)}$$ and

$$f_z' = \frac{f_z}{\gamma(v)\left(1 - \frac{\bar{v}\cdot\bar{u}}{c^2}\right)}$$

Finally, we show that $\bar{f} = \gamma m\bar{a} + \frac{\bar{f}\cdot\bar{u}}{c^2}\bar{u}$ (if $\dot{m} = 0$). Indeed,

$$\bar{f} = \frac{d\bar{p}}{dt} = \frac{d}{dt}(\gamma m\bar{u}) = \gamma m\bar{a} + m\bar{u}\dot{\gamma} =$$

$$= \gamma m\bar{a} + \frac{\bar{u}}{c^2}(mc^2\dot{\gamma}) = \gamma m\bar{a} + \frac{\bar{u}}{c^2}\dot{\mathcal{E}} =$$

$$= \gamma m\bar{a} + \frac{\bar{u}}{c^2}(\bar{f}\cdot\bar{u}).$$