OXFORD UNIVERSITY

PHYSICS DEPARTMENT

3RD YEAR UNDERGRADUATE COURSE

GENERAL RELATIVITY

PROBLEM SET 1

Solution notes

by

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Problem 1

Consider the following thought:

"Special relativity holds for frames moving at constant relative velocity, but of course acceleration requires general relativity because the frames are noninertial."

Ineffable twaddle. Special relativity certainly doesn't cower before simple kinematical acceleration. On the other hand, acceleration, even just uniform acceleration in one dimension, is not without its connections with general relativity. We shall explore some of them here. For ease of notation, we set $c = 1$*. In part (d) we'll put* c *back.*

1a) Let us first ask what we mean by "uniform acceleration." After all, a rocket approaching the speed of light c *can't change its velocity at a uniform rate forever without exceeding* c *at some point. Go into the frame moving instantaneously at velocity* v *with the rocket relative to the "lab." In this frame, by definition the instantaneous rocket velocity* $v\prime$ *vanishes. Wait a time* dt' *later, as measured in this frame. The rocket now has velocity* dv' *in this same frame. What we mean by constant acceleration is* $dv'/dt' \equiv a'$ *is constant. The acceleration measured in the fixed lab is certainly not constant! The question is, how is the lab acceleration* $a = dv/dt$ *related to the truly constant* a'*? To answer this, let* $V = v/\sqrt{1-v^2}$*, the spatial part of the 4-vector* V^α *associated with the ordinary velocity* v*. The same relation holds for* V' *and* v'*. Assume for the moment that the primed and unprimed frames differ by some arbitrary velocity* w*. The 4-velocity differentials are then given by:*

$$dV' = \left(dV - w dV^0\right) / \sqrt{1-w^2}\,.$$

Explain.

1b) Now, set $w = v$*. We thereby go into the frame in which* $v' = 0$*; the rocket is instantaneously at rest. Prove that* $dv = dv\prime(1 - v^2)$*. (Remember,* v *and* v' *are ordinary velocities.) From here, prove that*

$$\frac{dv}{dt} = a'(1-v^2)^{3/2}\,.$$

Solution:

This is the standard material of Special Relativity. For a particle moving in an interial frame S along the trajectory $x^\mu = x^\mu(\tau)$ parametrised by the proper time τ, one defines the 4-velocity $U^\mu = dx^\mu/d\tau$ and 4-acceleration $A^\mu = dU^\mu/d\tau$. Since $ds^2 = -c^2dt^2 + d\mathbf{x}^2 = -c^2d\tau^2$, we have $d\tau = dt\sqrt{1-\mathbf{v}^2/c^2} = dt/\gamma$, so $U^\mu = (\gamma c, \gamma\mathbf{v})$ and

$$A^\mu = \left(\gamma^4 \frac{\mathbf{v}\cdot\mathbf{a}}{c},\ \gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c^2}\mathbf{v} + \gamma^2\mathbf{a}\right)\,. \tag{1}$$

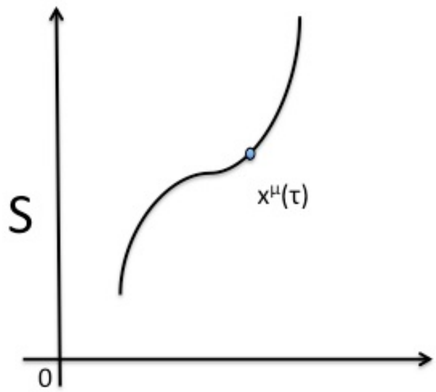

FIG. 1: A particle moving in an inertial frame S along the trajectory $x^\mu = x^\mu(\tau)$ parametrised by the proper time τ. The particle's 4-velocity U^μ is orthogonal (in Minkowski metric) to its 4-acceleration A^μ: $U_\mu A^\mu = 0$.

In particle's own frame, $A_0^\mu = (0, \mathbf{a}_0)$. Computing the invariant $A_\mu A^\mu = (A_0)_\mu A_0^\mu$, we find

$$\mathbf{a}_0^2 = \gamma^6 \frac{(\mathbf{v}\cdot\mathbf{a})^2}{c^2} + \gamma^4 \mathbf{a}^2 \,. \tag{2}$$

If $\mathbf{v}$ and $\mathbf{a}$ have the same direction, eq. (2) simplifies to $a_0^2 = a^2\gamma^6$. One can show that for a particle moving in S under the action of a constant force, $a_0 = const$. This is the situation considered in the present problem (we shall use the notation a_0 rather than a' in the solutions).

1c) Show that, starting from rest at $t = t' = 0$,

$$v = \frac{a't}{\sqrt{1 + a'^2 t^2}}\,, \qquad a't = \sinh(a't)\,,$$

and hence show that (for $x = 0$ at $t = t' = 0$):

$$v = \tanh a't'\,, \qquad x = \frac{1}{a'}\left[\cosh a't' - 1\right]\,.$$

The integrals are not difficult; do them yourselves.

Solution:

Since $a_0 = a\gamma^3 = const$ and $a = dv/dt$, we find

$$\int \frac{dv}{(1 - v^2/c^2)^{3/2}} = \int a_0 dt = a_0 t\,.$$

The integral on the LHS is equal to $v/\sqrt{1 - v^2/c^2}$ (it can be computed by changing variables $v \to \xi$, with $\xi^2 = v^2/(1 - v^2/c^2)$). Then

$$v(t) = \frac{a_0 t}{\sqrt{1 + a_0^2 t^2}}\,.$$

Since $v = dx/dt$, another integration gives

$$\int dx = a_0 \int \frac{tdt}{\sqrt{1 + a_0^2 t^2/c^2}}$$

or, with the initial condition $x(0) = 0$,

$$x(t) = \frac{c^2}{a_0} \left[\sqrt{1 + \frac{a_0^2 t^2}{c^2}} - 1 \right] . \tag{3}$$

Since $v(t)$ is known explicitly, we can compute $\tau(t)$. We have

$$\frac{d\tau}{dt} = \gamma^{-1} = \frac{1}{\sqrt{1 + \frac{a_0^2 t^2}{c^2}}}$$

and therefore

$$\tau = \int d\tau = \int \frac{dt}{\sqrt{1 + \frac{a_0^2 t^2}{c^2}}} = \frac{c}{a_0} \text{arcsinh} \left(\frac{a_0 t}{c} \right) .$$

(The integral above is taken with the trigonometric substitution, taken into account that $\cosh^2 z - \sinh^2 z = 1$.) Thus, we obtain, taking into account the result (3),

$$ct = \frac{c^2}{a_0} \sinh \left(\frac{a_0 \tau}{c} \right) , \tag{4}$$

$$x = \frac{c^2}{a_0} \left[\cosh \left(\frac{a_0 \tau}{c} \right) - 1 \right] . \tag{5}$$

The trajectory in the ct - x plane is a hyperbola,

$$\left(x + \frac{c^2}{a_0} \right)^2 - c^2 t^2 = \frac{c^4}{a_0^2} ,$$

centered at the origin, whose branches asymptote to $x = \pm ct - c^2/a_0$. Of course, we can choose a different initial condition for $x(t)$, e.g $x(0) = c^2/a_0$. Then

$$ct = \frac{c^2}{a_0} \sinh \left(\frac{a_0 \tau}{c} \right) , \tag{6}$$

$$x = \frac{c^2}{a_0} \cosh \left(\frac{a_0 \tau}{c} \right) , \tag{7}$$

and the hyperbola $x^2 - c^2 t^2 = c^4/a_0^2$ is asymptotic to the light-cone $x = \pm ct$. The motion with constant proper acceleration is therefore sometimes called the hyperbolic motion (see fig. 2).

Note also that

$$\frac{v}{c} = \tanh \frac{a_0 \tau}{c} .$$

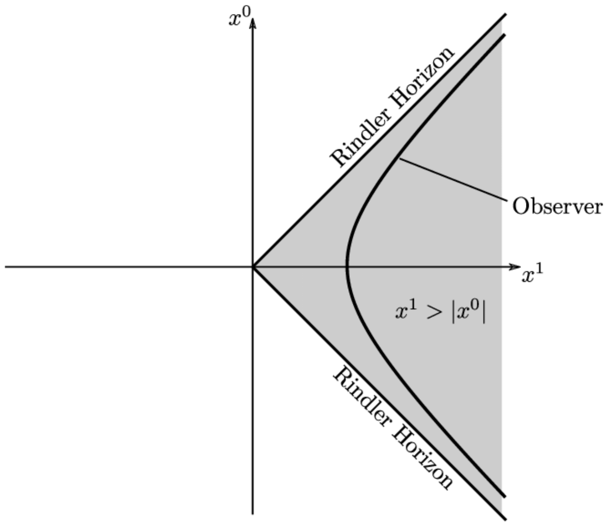

FIG. 2: The Rindler wedge. The accelerated observer is moving along the hyperbola (here normalised as in Eqs. (6), (7)).

1d) Let's use these results to construct a full coordinate transformation from the lab frame (x,t) to the accelerating (x',t') frame. A good start is to guess a transform of the form $t = A(x')\sinh(a't') + B(x')$, $x = A(x')\cosh(a't') + C(x')$, where A, B, and C depend only upon x'. Then on $x' = constant$ surfaces, $dx/dt = tanh(a't') = v$, which is indeed what we need.

By definition, constant t' surfaces are constant time surfaces in the (x',t') frame that moves instantaneously with velocity $v = \tanh(a't')$ with respect to the (x,t) frame. On such a surface, $dt'/dx' = 0$. We fix the origin by demanding that as $t' \to 0$, $x \to x'$. We fix our clock by demanding that as $t' \to 0$, $t \to t'$ at the rocket location $x' = 0$. (This must be done locally: since A depends on x', this time agreement can be exact at only one value of x'.) Show that these constraints force B and C to be constant, and that B in particular must vanish. Finally, put the speed of light c back into the equations, demand that x goes to x' at $t' = 0$, and show that

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh\frac{a't'}{c}\,, \qquad x = \left(\frac{c^2}{a'} + x'\right)\cosh\frac{a't'}{c} - \frac{c^2}{a'}\,, \tag{8}$$

Solution:

Since the accelerated frame S' is not an inertial frame, the transformation (x,t) to (x',t') is not a Lorentz transformation. There is a general method of constructing a transformation to a local coordinate system associated with an arbitrarily moving observer (in flat or curved space-time). This method is systematic and does not involve any guesses. We shall split the answer to 1d) into two parts: first, in Part A, we shall proceed as suggested in 1d); then, in Part B, we shall provide a more systematic approach and relevant references.

Part A

For $t' = const$, we have $dt' = 0$. But $dt' = \gamma\,(dt - vdx)$ (this is a Lorentz transformation to an instantaneous comoving frame), so $dt - vdx = 0$. Using $t = A(x')\sinh{(a't')} + B(x')$ and $x = A(x')\cosh{(a't')} + C(x')$ with $t' = const$ and $v = \tanh{(a't')}$, we find $\partial_{x'}B(x') = \partial_{x'}C(x')\tanh{(a't')}$. Since B and C depend only on x', for generic $t' = const$ this can be satisfied only if $B = const$ and $C = const$.

Then consider the limit $t' \to 0$. In this limit, $t = A(x')a't' + B(x')$. We demand $t \to t'$ as $t' \to 0$ at the rocket location $x' = 0$, so $B(0) = 0$ (and hence $B = 0$) and $A(0) = -1/a'$. In the same limit $t' \to 0$, $x = A(x') + C(x')$, and we demand $x \to x'$, thus $x' = A(x') + C(x')$ at $t' \to 0$ but also for any t' since this relation is independent of t'. This gives $C(x') = x' - A(x')$, and in the limit $x' \to 0$ we have $C(0) = -A(0) = -1/a'$. But $C = const$, so $C = -1/a'$. Finally, $A(x') = x' - C = x' + 1/a'$. Restoring c, we find

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh\frac{a't'}{c}\,, \qquad x = \left(\frac{c^2}{a'} + x'\right)\cosh\frac{a't'}{c} - \frac{c^2}{a'}\,. \tag{9}$$

Part B

More systematically, at any fixed pont O' on the hyperbola (the worldline of the particle experiencing a hyperbolic motion in S) we need to choose a basis of four 4-vectors $\mathbf{e}^{\mu}_{(k)}$ (here $k = 0, 1, 2, 3$ label the vectors, whereas μ denotes their components in S; vectors $\mathbf{e}^{\mu}_{(0)}$, $\mathbf{e}^{\mu}_{(1)}$, $\mathbf{e}^{\mu}_{(2)}$, $\mathbf{e}^{\mu}_{(3)}$ are thus the 4-dimensional analogs of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ in the usual Euclidean three-dimensional space: for example, $\mathbf{i}$ has coordinates $(1, 0, 0)$, and so on). It is natural to choose the proper time τ as t' at O' and U^{μ}/c as $\mathbf{e}^{\mu}_{(0)}$, where $u^{\mu} = dx^{\mu}/d\tau$ is the 4-velocity. In the instantaneous comoving frame, $U^{\mu} = (c, \mathbf{0})$, whereas in S we have

$$\mathbf{e}^{\mu}_{(0)} = \left(\cosh\left[\frac{a_0\tau}{c}\right], \sinh\left[\frac{a_0\tau}{c}\right], 0, 0\right)\,.$$

The other 4-vectors, $\mathbf{e}^{\mu}_{(1)}$, $\mathbf{e}^{\mu}_{(2)}$, $\mathbf{e}^{\mu}_{(3)}$, should be orthogonal to $\mathbf{e}^{\mu}_{(0)}$ in Minkowski metric, $\eta_{\mu\nu}\mathbf{e}^{\mu}_{(i)}\mathbf{e}^{\nu}_{(k)} = \eta_{ik}$. In particular, we should have $\eta_{\mu\nu}\mathbf{e}^{\mu}_{(0)}\mathbf{e}^{\nu}_{(1)} = 0$. Since $U^{\mu}A_{\mu} = 0$, a natural candidate for $\mathbf{e}^{\mu}_{(1)}$ is a 4-vector proportional to the 4-acceleration A^{μ}:

$$\mathbf{e}^{\mu}_{(1)} = \left(\sinh\left[\frac{a_0\tau}{c}\right], \cosh\left[\frac{a_0\tau}{c}\right], 0, 0\right)\,.$$

The other two 4-vectors are "spectators". They are given by

$$\mathbf{e}^{\mu}_{(2)} = (0, 0, 1, 0)\ ,$$

$$\mathbf{e}^{\mu}_{(3)} = (0,0,0,1)\ .$$

For a generic point P in S characterised by the position 4-vector x^μ, one can write

$$x^\mu = x_0^\mu + x'^k \mathbf{e}^{\mu}_{(k)}\,, \tag{10}$$

where $x_0^\mu = (ct, x, 0, 0)$, with ct and x given by eqs. (6), is the position of the point O' moving along the hyperbola (i.e. the position of the origin of the new coordinate system S'), and $x'^k \mathbf{e}^{\mu}_{(k)}$ is the position of P in S'. Eqs. (10) are 4 equations for 4 unknowns (x'^k). For O', we already identified $x'^0 = ct' = c\tau$, and we know that $x'^2 = x^2$ and $x'^3 = x^3$ are "spectators". For x'^1 we have

$$x^0 = \left(\frac{c^2}{a_0} + x'^1\right)\sinh\left(\frac{a_0 x'^0}{c^2}\right), \tag{11}$$

$$x^1 = \left(\frac{c^2}{a_0} + x'^1\right)\cosh\left(\frac{a_0 x'^0}{c^2}\right) - \frac{c^2}{a_0}\,. \tag{12}$$

The new coordinates are local - they cover only part of Minkowski space known as the Rindler wedge. These issues are discussed in detail in the book by Misner, Thorn and Wheeler, "Gravity", sections 6.2 - 6.6 and 13.6.

A good video explaining various aspects of hyperbolic motion including the coordinate system of an accelerated observer is available at

`https://www.youtube.com/watch?v=yyzPCtml158&list=PLJHszsWbB6hqlw73QjgZcFh4DrkQLSCQa&index=16` (Part I) and

`https://www.youtube.com/watch?v=O92pQXZaEnw` (Part II).

1e) Show that the invariant Minkowski line element may be written in (x', t') coordinates as:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 = \left(1 + \frac{x'a'}{c^2}\right)^2 c^2 dt'^2 - dx'^2\,.$$

Provide a physical interpretation of your result in terms of a gravitational redshift. How do you interpret the region $x' \leq -c^2/a'$? (Review the results of 1d.)

Solution:

The line element $ds^2 = -c^2dt^2 + dx^2$ is computed in coordinates (x'^0, x'^1) by a direct substitution $(x \equiv x'^1)$, using eqs. (11), (12). The metric tensor in S' is not the Minkowski metric tensor (except at $x' = 0$). The frame S' is not an inertial frame, so this is expected. The proper time and the coordinate time are related by

$$d\tau = \left(1 + \frac{x'a'}{c^2}\right) dt'$$

which is similar to the gravitational redshift. In S', only part (more precisely, 1/4) of the Minkowski space-time is properly covered by the new coordinates. Since

$$\left(x + \frac{c^2}{a_0}\right)^2 - c^2t^2 = \left(\frac{c^2}{a_0} + x'\right)^2 ,$$

the condition $x' = -c^2/a_0$ corresponds to the light-cone lines $x + \frac{c^2}{a_0} = \pm ct$. The new coordinates cover the interior of this region, with $x' > -c^2/a_0$, known as the *Rindler wedge*.

Problem 2 (GR, 2022)

Rel. particle of mass m ($m=1$ later) moving in potential $U = -\alpha/R$:

$$\begin{cases} \gamma c^2 - \alpha/R = E \\ \gamma R^2 \dot{\varphi} = J \end{cases}$$

- see SR for proof
- polar coord. R, φ used

2a) This is cons. of energy and angular mom. – see SR (B2).

2b) $$\frac{dR}{d\varphi} = \dot{R}/\dot{\varphi}$$

Now, $$\dot{\varphi} = J/\gamma R^2 = \frac{Jc^2}{R^2(E+\alpha/R)}.$$

Also, $$\gamma^2 c^4 = (E+\alpha/R)^2$$

$$\gamma^2 = (1-\bar{v}^2/c^2)^{-1}$$ Recall: $\bar{v} = (\dot{R}, R\dot{\varphi})$ in polar coord.

$$1-\frac{\bar{v}^2}{c^2} = c^4\Big/\left(E+\alpha/R\right)^2$$

$$1-\frac{\dot{R}^2}{c^2}-\frac{R^2\dot{\varphi}^2}{c^2} = \frac{c^4}{(E+\alpha/R)^2}$$

$$\frac{\dot{R}^2}{c^2} = 1-\frac{R^2}{c^2}\,\frac{J^2c^4}{R^4(E+\alpha/R)^2}-\frac{c^4}{(E+\alpha/R)^2} =$$

$$= \frac{1}{(E+\alpha/R)^2}\left[\left(E+\frac{\alpha}{R}\right)^2-\frac{J^2c^2}{R^2}-c^4\right]$$

$$\frac{\dot{R}^2}{\dot{\varphi}^2} = \frac{R^4}{J^2c^4}\,c^2\left[\left(E+\frac{\alpha}{R}\right)^2-\frac{J^2c^2}{R^2}-c^4\right] =$$

$$= \frac{R^4}{J^2c^2}\left[E^2-c^4+\frac{2\alpha E}{R}+\frac{\alpha^2-J^2c^2}{R^2}\right].$$

2c) Let $u = 1/R$. Then $du = -\frac{dR}{R^2}$

$$\frac{dR}{d\varphi} = -\frac{R^2 du}{d\varphi}$$

$$\left(\frac{dR}{d\varphi}\right)^2 = R^4\left(\frac{du}{d\varphi}\right)^2 = \frac{1}{u^4}\left(u'_\varphi\right)^2 =$$

$$= \frac{1}{u^4 J^2 c^2}\left[E^2 - c^4 + 2\alpha E u + \left(\alpha^2 - J^2 c^2\right)u^2\right]$$

$$\Rightarrow \boxed{\left(u'_\varphi\right)^2 = \frac{E^2 - c^4}{J^2 c^2} + \frac{2\alpha E}{J^2 c^2}u + \frac{\alpha^2 - J^2 c^2}{J^2 c^2}u^2}$$

Differentiate w.r.t. φ:

$$2u'_\varphi u''_{\varphi\varphi} = \frac{2\alpha E}{J^2 c^2}u'_\varphi + \left(\frac{\alpha^2}{J^2 c^2} - 1\right)2uu'_\varphi$$

$u'_\varphi \neq 0 \Rightarrow$

$$\boxed{u''_{\varphi\varphi} - \left(\frac{\alpha^2}{J^2 c^2} - 1\right)u = \frac{\alpha E}{J^2 c^2}}$$

Inhomogeneous linear ODE

$$u''_{\varphi\varphi} + \mu^2 u = \alpha E / J^2 c^2 ,$$

$$\mu^2 \equiv 1 - \alpha^2 / J^2 c^2$$

2d) $u = A \cos \mu\varphi + B$: solution.

$$\Rightarrow \quad u = \frac{\alpha E}{\mu^2 J^2 c^2} \left(1 + \varepsilon \cos \mu\varphi \right)$$

with obvious identifications.

$$\Rightarrow \boxed{ R(\varphi) = \frac{\mu^2 J^2 c^2 / \alpha E}{1 + \varepsilon \cos \mu\varphi} }$$

This is ellipse in polar coord. (why?) but the orbit is not closed and has a period of $2\pi/\mu$ (see LL vol 1).

$$\mu \simeq 1 - \alpha^2 / 2 J^2 c^2 + \ldots$$

$$T \simeq 2\pi - \pi\alpha^2 / J^2 c^2 + \ldots \qquad \Delta\varphi = \frac{\pi G^2 M^2}{J^2 c^2}$$

$\longrightarrow$ perihelion advance

Problem 2

Recognising tensors. One way to prove that something is a vector or tensor is to show explicitly that it satisfies the coordinate transformation laws. This can be a long and arduous procedure if the tensor is complicated, like $R^{\lambda}_{\mu\nu\kappa}$. There is another way, usually much better! Show that if V_ν is an arbitrary covariant vector and the combination $T^{\mu\nu}V_\nu$ is known to be a contravariant vector (note the free index μ), then

$$\left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\right)V'_{\nu} = 0$$

Why does this prove that $T_{\mu\nu}$ is a tensor? Does your proof actually depend on the rank of the tensors involved?

Solution:

If $T^{\mu\nu}V_\nu$ and V_ν are known to be tensors (vectors), we can write

$$T'^{\mu\nu}V'_{\nu} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}}T^{\lambda\kappa}V_{\kappa} = \frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\kappa}}T^{\lambda\kappa}V'_{\nu}$$

which implies

$$\left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\right)V'_{\nu} = 0$$

and

$$T'^{\mu\nu} = T^{\lambda\sigma}\frac{\partial x'^{\mu}}{\partial x^{\lambda}}\frac{\partial x'^{\nu}}{\partial x^{\sigma}}\,,$$

since V_ν is arbitrary. The same reasoning applies to tensors of arbitrary rank.

Problem 3

What about $d^2x_\mu/d\tau^2$? The geodesic equation in standard form gives us an expresssion for $d^2x_\mu/d\tau^2$ in terms of the affine connection, $\Gamma^\mu_{\lambda\sigma}$. For the covariant coordinate x_μ, show that

$$\frac{d^2x_\mu}{d\tau^2} = \frac{1}{2}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\frac{\partial g_{\nu\rho}}{\partial x^\mu}$$

Refer to section 4.7 in the notes if help is needed. Under what conditions is $dx_0/d\tau = V_0 \equiv V_t$ a constant of the motion?

Solution:

The geodesic equation is

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\,,$$

where the dot denotes the derivative w.r.t. τ (or any affine parameter). It can also be written in the form

$$\frac{DV^\mu}{D\tau} = 0\,,$$

where $V^\mu = dx^\mu/d\tau$ is the 4-velocity. Since in GR the connection is compatible with the metric, i.e. the covariant derivative of the metric tensor vanishes, we can write

$$\frac{D(g^{\mu\nu}V_\nu)}{D\tau} = g^{\mu\nu}\frac{DV_\nu}{D\tau} = 0\,,$$

i.e.

$$\frac{DV_\mu}{D\tau} = \frac{dV_\mu}{d\tau} - \Gamma^\rho_{\sigma\mu}V^\sigma V_\rho = 0\,.$$

Explicitly, since

$$\Gamma^\rho_{\sigma\mu} = \frac{g^{\rho\kappa}}{2}\left(\partial_\sigma g_{\mu\kappa} + \partial_\mu g_{\sigma\kappa} - \partial_\kappa g_{\sigma\mu}\right)\,,$$

we find

$$\Gamma^\rho_{\sigma\mu}V^\sigma V_\rho = \frac{1}{2}\left(\partial_\sigma g_{\mu\kappa} + \partial_\mu g_{\sigma\kappa} - \partial_\kappa g_{\sigma\mu}\right)V^\sigma V^\kappa = \frac{1}{2}\partial_\mu g_{\sigma\kappa}V^\sigma V^\kappa\,,$$

where the last equality follows from contracting the symmetric and antisymmetric pieces. So, we have

$$\frac{dV_\mu}{d\tau} = \frac{1}{2}\partial_\mu g_{\sigma\kappa}V^\sigma V^\kappa\,, \tag{13}$$

which is the same as

$$\frac{d^2x_\mu}{d\tau^2} = \frac{1}{2}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\frac{\partial g_{\nu\rho}}{\partial x^\mu}\,,$$

remembering that $V_\mu = dx_\mu/d\tau$ and changing the dummy indices.

From Eq. (13) it follows that V_0 is a constant of motion if the metric is static (time-independent).

Problem 4

4a) Practise with the Ricci Tensor. Consider the 2D surface given by $z^2 = x^2 + y^2$ *where* x, y, z *are Cartesian coordinates in 3D Euclidian space. This represents a pair of cones centred on the origin, one cone opening upward, the other opening downward. The opening angle is* 45° *measured from the* z *axis. Justify this description.*

Solution:

At each fixed z, this is a circle of radius $R = z$. So there are 2 cones meeting at the origin and opening up in two directions at the angle of 45° measured from the z axis.

4b) A point in the 2D conic surface can be determined by R*, the cylindrical radius of the point measured from the z-axis, and* ϕ*, the usual azimuthal angle. Show that the metric for the 2D surface in these coordinates is*

$$ds^2 = 2dR^2 + R^2 d\phi^2 \,.$$

Solution:

The embedding of the surface into $\mathbb{R}^3$ is given by the equation $z^2 = x^2 + y^2$, or $z = \pm R$ in cylindrical coordinates. The Euclidean metric in $\mathbb{R}^3$ in cylindrical coordinates is

$$ds^2 = dR^2 + R^2 d\phi^2 + dz^2 \,.$$

With $dz = \pm dR$, we get

$$ds^2 = 2dR^2 + R^2 d\phi^2 \,.$$

4c) Is this 2D surface curved, in the mathematical sense of having nonvanishing components of the curvature tensor $\mathcal{R}_{\lambda\kappa\mu\nu}$*? (We use* $\mathcal{R}$ *for the tensor,* R *for the radial coordinate.) Answer the question by showing that the metric of part 4b) can be transformed to new coordinates* R', ϕ'*, for which* $ds^2 = dR^2 + R^2 d\phi^2$*. (The transformation law is extremely simple!) Why does this result alone answer the posed question? Can you give a physical interpretation of your mathematical transformation?*

Solution:

A simple rescaling $R' = \sqrt{2}R$, $\phi' = \phi/\sqrt{2}$ bring the metric on the 2D surface to the form

$$ds^2 = dR'^2 + R'^2 d\phi'^2$$

which is just a 2D flat space. Therefore, the surface is not curved, all components of the curvature tensor $\mathcal{R}_{\lambda\kappa\mu\nu}$ vanish. You can make a cone or a cylinder out of a flat sheet of paper without any distortions (you cannot make a sphere this way).

4d) Next, consider a different 2D surface: $z = (\alpha/2)(x^2 + y^2)$, where α is an arbitrary constant parameter. Show that this is a paraboloid of revolution, i.e. a parabola spun around the z-axis. Prove that the metric within this surface is given by

$$ds^2 = (1 + \alpha^2 R^2)dR^2 + R^2 d\phi^2 \,.$$

Solution:

In cylindrical coordinates, the embedding equation is $z = \alpha/2R^2$, so $dz = \alpha R dR$. Then the metric

$$ds^2 = dR^2 + R^2 d\phi^2 + dz^2$$

becomes

$$ds^2 = (1 + \alpha^2 R^2)dR^2 + R^2 d\phi^2$$

on the embedded surface.

4e) Prove that this surface is distorted by curvature. Calculate, for example, $\mathcal{R}_{\phi\phi}$ and show that it is not zero, but given by

$$\mathcal{R}_{\phi\phi} = -\frac{\alpha^2 R^2}{(1 + \alpha^2 R^2)^2} \,.$$

You should show en route that the only non-vanishing affine connection coefficients are

$$\Gamma^R_{RR} = \frac{\alpha^2 R}{1 + \alpha^2 R^2} \,, \qquad \Gamma^\phi_{\phi R} = \Gamma^\phi_{R\phi} = \frac{1}{R} \,, \qquad \Gamma^R_{\phi\phi} = -\frac{R}{1 + \alpha^2 R^2} \,.$$

Solution:

This is a direct calculation.

A simple way to compute the Christoffel symbols (the affine connection coefficients) is to strat with the metric

$$ds^2 = (1 + \alpha^2 R^2)dR^2 + R^2 d\phi^2$$

and form a Lagrangian

$$L = g_{\mu\nu}\dot{x^\mu}\dot{x^\nu} = g_{RR}\dot{R}^2 + g_{\phi\phi}\dot{\phi}^2 = (1 + \alpha^2 R^2)\dot{R}^2 + R^2\dot{\phi}^2 \,,$$

where the dot denotes the derivative w.r.t. an affine parameter λ (e.g. the proper time), compare the corresponding Euler-Lagrange equations,

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} = \frac{\partial L}{\partial x^\mu} \,,$$

with the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\sigma}\dot{x}^\nu\dot{x}^\sigma = 0\,,$$

and read off the $\Gamma^\mu_{\nu\sigma}$.

Here we have for $\mu = R$:

$$\frac{\partial L}{\partial \dot{R}} = 2(1+\alpha^2 R^2)\dot{R}$$

and then

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{R}} = 2(1+\alpha^2 R^2)\ddot{R} + 4\alpha^2 R\dot{R}^2\,,$$

whereas

$$\frac{\partial L}{\partial R} = 2\alpha^2 R\dot{R}^2 + 2R\dot{\phi}^2\,.$$

Then the Euler-Lagrange equation reads

$$\ddot{R} + \frac{\alpha^2 R}{1+\alpha^2 R^2}\,\dot{R}^2 - \frac{R}{1+\alpha^2 R^2}\dot{\phi}^2 = 0\,,$$

and a comparison with the geodesic equation immediately gives

$$\Gamma^R_{RR} = \frac{\alpha^2 R}{1+\alpha^2 R^2}\,, \qquad \Gamma^R_{\phi\phi} = -\frac{R}{1+\alpha^2 R^2}\,.$$

Similarly, one finds the connection coefficients with $\mu = \phi$.

There are various formulas one can use to compute the components of the Ricci tensor (for diagonal metrics, these formulas are simpler). They are given in Lecture notes and other sources. The result is

$$\mathcal{R}_{\phi\phi} = -\frac{\alpha^2 R^2}{(1+\alpha^2 R^2)^2}\,.$$

Problem 5

What is "the spatial part" of a metric? It is easy, even trivial, to get the proper time from a metric. One simply sets all the spatial $dx^i = 0$ in the invariant interval $g_{\mu\nu} dx^\mu dx^\nu$, and reads off a proper time of

$$d\tau = \sqrt{-g_{00}}dx^0/c\,.$$

This is what a local inertial observer reads off on their watch. So to get " the spatial part" of the metric, call it dl^2, do we just take whatever is left over from setting $dx^0 = 0$, i.i. $dl^2 = g_{ij}dx^i dx^j$? Not quite.

How does an observer actually measure a distance? They take a light ray, bounce it off a mirror a distance dl away, measure the (proper) time on their watch $d\tau$ for the light to go and come back, and then set $dl = c d\tau/2$. Let's go with that.

5a) Show that for a diagonal metric tensor (all $g_{0i} = g_{i0} = 0$), this procedure gives $dl^2 = g_{ij}dx^i dx^j$, just as we expect.

Solution:

The geodesic of a photon is $ds^2 = 0$ (the light cone). For a diagonal metric, this implies

$$\sqrt{-g_{00}}dx^0 = \pm\sqrt{g_{ij}dx^i dx^j}\,,$$

depending on the direction of propagation. In the measurement described, a photon travels to the mirror and back, so

$$d\tau = \frac{2}{c}\sqrt{g_{ij}dx^i dx^j}\,,$$

and then indeed $dl = c d\tau/2$ gives $dl = \sqrt{g_{ij}dx^i dx^j}$.

5b) Show that for a general metric tensor $g_{\mu\nu}$, with $g_{0i} = g_{i0}$ present, this procedure gives $dl^2 = \gamma_{ij}dx^i dx^j$, where $\gamma_{ij} = g_{ij} - (g_{0i}g_{0j}/g_{00})$.

The metric tensor of a rotating black hole (the Kerr metric) actually has $g_{0\phi} = g_{\phi 0}$ components, so this formula is very relevant here. We see that the spatial part of the metric may contain mixed time-indexed terms!

Solution:

Now the null geodesic is a quadratic equation for dx^0:

$$g_{00}(dx^0)^2 + 2g_{0i}dx^0 dx^i + g_{ij}dx^i dx^j = 0\,.$$

Solving this equation gives

$$dx^0 = -\frac{g_{0i}}{g_{00}}dx^i \pm \frac{1}{g_{00}}\sqrt{(g_{0i}g_{0j} - g_{00}g_{ij})dx^i dx^j}$$

and

$$cd\tau_{\mp} = \frac{g_{0i}}{\sqrt{-g_{00}}}dx^i \mp \sqrt{(g_{ij} - g_{0i}g_{0j}/g_{00})dx^i dx^j}\,,$$

where the $\pm$ sign reflects the direction of velocity. The total length ("back and forth") is $dl = (cd\tau_+ - cd\tau_-)/2$, i.e.

$$dl = \sqrt{(g_{ij} - g_{0i}g_{0j})dx^i dx^j}\,,$$

and the result follows.

5c) Using the $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$ relations, show that

$$g^{ij}\gamma_{jk} = \delta^i_k$$

and that, defining γ^{ij} by raising indices via $\gamma^{ij} \equiv g^{ik}g^{jm}\gamma_{km}$, leads to

$$\gamma^{ij} = g^{ij}\,,$$

the "pure spatial part" of $g_{\mu\nu}$. The matrix γ^{ij} defined this way is indeed the inverse of γ_{ij}. Hence, we are justified in regarding $\gamma_{ij}dx^i dx^j$ as the invariant interval in its own three-dimensional space, with inverse γ^{ij}, within the more encompassing four-dimensional $g_{\mu\nu}$ spacetime. (Note that this also shows that the indices on γ_{ij} may be raised with γ^{ij}.)

Solution:

We need to compute $g^{ij}\gamma_{jk}$ which is

$$g^{ij}\gamma_{jk} = g^{ij}\left(g_{jk} - g_{0j}g_{0k}/g_{00}\right)\,. \tag{14}$$

From $g^{\mu\nu}g_{\nu\rho} = \delta^\mu_\rho$ we get, first by setting $\mu = i$ and $\rho = k$,

$$g^{i0}g_{0k} + g^{ij}g_{jk} = \delta^i_k\,,$$

and then by setting $\mu = i$ and $\rho = 0$,

$$g^{i0}g_{00} + g^{ij}g_{j0} = 0\,.$$

Using the ingredients for $g^{ij}g_{jk}$ and $g^{ij}g_{j0}$ from these equations in Eq. (14), we find $g^{ij}\gamma_{jk} = \delta^i_k$.

Clearly then, $\gamma^{ij} \equiv g^{ik}g^{jm}\gamma_{km} = g^{ik}\delta^j_k = g^{ij}$.

5d) Show that $\det g_{\mu\nu} = g_{00}\,\det\gamma_{ij}$, which is consistent with identifying γ_{ij} as the spatial metric. You may find it useful to recall that the determinant of a matrix is unchanged when a multiple of one row is added to another.

Solution:

We start with

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{15}$$

One can add to the second, third and forth rows of $g_{\mu\nu}$ the top row multiplied, respectively, by $-g_{i0}/g_{00}$, $i = 1, 2, 3$. This will make the first column zero except for g_{00}:

$$det\ g_{\mu\nu} = det \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ 0 & g_{11} - \frac{g_{10}g_{01}}{g_{00}} & g_{12} - \frac{g_{10}g_{02}}{g_{00}} & g_{13} - \frac{g_{10}g_{03}}{g_{00}} \\ 0 & g_{21} - \frac{g_{20}g_{01}}{g_{00}} & g_{22} - \frac{g_{20}g_{02}}{g_{00}} & g_{23} - \frac{g_{20}g_{03}}{g_{00}} \\ 0 & g_{31} - \frac{g_{30}g_{01}}{g_{00}} & g_{32} - \frac{g_{30}g_{02}}{g_{00}} & g_{33} - \frac{g_{30}g_{03}}{g_{00}} \end{pmatrix}. \tag{16}$$

By construction, the spatial entries are γ_{ij}:

$$det\ g_{\mu\nu} = det \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ 0 & \gamma_{11} & \gamma_{12} & \gamma_{13} \\ 0 & \gamma_{21} & \gamma_{22} & \gamma_{23} \\ 0 & \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}. \tag{17}$$

Then, $det\, g_{\mu\nu} = g_{00}\, det \gamma_{ij}$.