OXFORD UNIVERSITY
PHYSICS DEPARTMENT
3RD YEAR UNDERGRADUATE COURSE

GENERAL RELATIVITY

PROBLEM SET 2

EQUILIBRIUM, FLOWS, AND ORBITS IN GENERAL RELATIVITY

Solution notes

by

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Problem 1

Hydrostatic Equilibrium in GR. Model a neutron star atmosphere with a simple equation of state: $P = K\rho^\gamma$, where P is pressure, ρ is mass density, γ is the adiabatic index and K is a constant. Assume that $g_{00} = -(1 - 2GM/rc^2)$, where M is the mass of the star and r is radius. If $\rho = \rho_0$ at the surface $r = R_0$, solve the equation of hydrostatic equilibrium to show that

$$\frac{1 + K\rho^{\gamma-1}/c^2}{1 + K\rho_0^{\gamma-1}/c^2} = \left(\frac{1 - R_S/r_0}{1 - R_S/r}\right)^\alpha ,$$

where $R_S = 2GM/c^2$ is the so-called Schwarzschild radius, and $2\alpha\gamma = \gamma - 1$. (Hint: See § 4.6 of the notes.) What is the Newtonian limit of the above equation? Express your answer in terms of the speed of sound a, $a^2 = \gamma P/\rho$ and the potential $\Phi(r) = -GM/r$. (OPTIONAL: For those who have studied fluids, what quantity is being conserved in the Newtonian limit?)

Solution: Here we encounter fluid dynamics in curved space-time. Recall that fluid dynamics is the dynamics of densities of conserved charges. In the relativistic case, these are T^{00} and T^{0i} components of the conserved energy-momentum tensor. In flat space, the conservation law is simply

$$\partial_\mu T^{\mu\nu} = 0 . \tag{1}$$

Other components of $T^{\mu\nu}$ are related to T^{00} and T^{0i} via the constitutive relations

$$T^{\mu\nu} = P\eta^{\mu\nu} + \left(\rho + P/c^2\right) u^\mu u^\nu + \cdots , \tag{2}$$

where the infinite tail involving derivatives of $u^\mu(x)$ is omitted (we assume here that the gradients of $u^\mu(x)$ are small which of course may not always be the case). This (covariant) form of $T^{\mu\nu}$ follows either by applying Lorentz transformation (with the velocity encoded in $u^\mu = (\gamma c, \gamma \mathbf{v})$) to $T_0^{\mu\nu}$ in fluid's rest frame,

$$T_0^{\mu\nu} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} , \tag{3}$$

or by writing the most general covariant expression involving all the relevant ingredients ($\eta_{\mu\nu}$ and u^μ - but not $\partial_\nu u^\mu$ if we assume small gradients),

$$T^{\mu\nu} = A\,\eta^{\mu\nu} + Bu^\mu u^\nu ,$$

and then comparing this expression in fluid's rest frame (where $u^\mu = (c, \mathbf{0})$) to eq. (3) to read off the coefficients A and B.

In curved space-time, eqs. (1) and (2) are replaced by

$$T^{\mu\nu} = P g^{\mu\nu} + \left(\rho + P/c^2\right) u^\mu u^\nu + \cdots\,, \tag{4}$$

$$\nabla_\mu T^{\mu\nu} = 0\,, \tag{5}$$

where $g_{\mu\nu} u^\mu u^\nu = -c^2$. Explicitly, the equation (5) with $\nu = 0$ is

$$\frac{\partial P}{\partial x^\mu} + \left(\rho c^2 + P(\rho)\right) \frac{\partial \ln |g_{00}|^{1/2}}{\partial x^\mu} = 0 \tag{6}$$

which is supposed to be supplemented by the equation of state $P = P(\rho)$.

In this problem, the equation of state is assumed to be of the form $P = K\rho^\gamma$. With $\mu = r$, Eq. (6) is

$$\frac{\partial P}{\partial r} + \left(\rho c^2 + P(\rho)\right) \frac{\partial \ln |g_{00}|^{1/2}}{\partial r} = 0\,, \tag{7}$$

which can be integrated immediately:

$$\int \frac{dP}{\rho c^2 + P} = -\int d\left(\ln |g_{00}|^{1/2}\right)$$

gives

$$\int \frac{P'_\rho\, d\rho}{\rho c^2 + P} + \ln |g_{00}|^{1/2} = const\,.$$

Substituting $P = K\rho^\gamma$, we find

$$\frac{K\gamma}{c^2(\gamma-1)} \int \frac{d\rho^{\gamma-1}}{1 + K\rho^{\gamma-1}/c^2} + \ln |g_{00}|^{1/2} = const\,.$$

This gives

$$\ln\left(1 + \frac{K}{c^2}\rho^{\gamma-1}\right) + \frac{\gamma-1}{2\gamma} \ln\left(1 - \frac{2GM}{c^2 r}\right) = const\,,$$

where we used the explicit expression for g_{00} of the Schwarzschild metric[1]. We can also re-write the above expression as

$$\left(1 + \frac{K}{c^2}\rho^{\gamma-1}\right)\left(1 - \frac{2GM}{c^2 r}\right)^{\frac{\gamma-1}{2\gamma}} = const\,.$$

[1] Note that we tacitly assume here that the metric remains unaffected by the atmosphere of the neutron star, i.e. the metric is still the Schwarzschild metric despite the presence of relativistic matter rather than vacuum in the region outside the star.

The constant is fixed by the condition that $\rho = \rho_0$ at $r = R_0$. Then

$$\boxed{\frac{1+\frac{K}{c^2}\rho^{\gamma-1}}{1+\frac{K}{c^2}\rho_0^{\gamma-1}} = \left(\frac{1-\frac{2GM}{c^2R_0}}{1-\frac{2GM}{c^2r}}\right)^{\frac{\gamma-1}{2\gamma}}}$$

Taking the Newtonian limit involves expanding the above expression in the limit $c \to \infty$, or, more precisely, $K\rho^{\gamma-1}/c^2 \ll 1$ and $GM/c^2r \ll 1$. This gives

$$1 + \frac{K}{c^2}\rho^{\gamma-1} - \frac{K}{c^2}\rho_0^{\gamma-1} + \cdots = 1 - \frac{\gamma-1}{\gamma}\frac{GM}{c^2R_0} + \frac{\gamma-1}{\gamma}\frac{GM}{c^2r} + \cdots . \tag{8}$$

For the speed of sound we have $v_s^2 = \partial P/\partial\rho = K\gamma\rho^{\gamma-1} = \gamma P/\rho$. Then Eq. (8) gives

$$\frac{v_s^2}{\gamma-1} + \Phi(r) = \frac{v_{s,0}^2}{\gamma-1} + \Phi(R_0)\,,$$

where $\Phi(r) = -GM/r$ is the Newtonian gravitational potential.

Problem 2

Bondi Accretion: go with the flow. To get some practise working with the equations of GR as well as some insight into relativistic dynamics in a practical problem in astrophysics, consider what is known as (relativistic) Bondi Accretion, the spherical flow of gas into a black hole. (The original Bondi accretion problem was Newtonian accretion onto an ordinary star.) We assume a Schwarzschild metric in the usual spherical coordinates:

$$g_{00} = -\left(1 - 2GM/rc^2\right)\,, g_{rr} = \left(1 - 2GM/rc^2\right)^{-1}\,, g_{\theta\theta} = r^2\,, g_{\phi\phi} = r^2 \sin^2\theta\,.$$

2a) First, let us assume that particles are neither created or destroyed. So particle number is conserved. If n is the particle number density in the local rest frame of the flow, then the particle flux is $J^\mu = nU^\mu$, where U^μ is the flow 4-velocity. Justify this statement, and using §4.5 in the notes, show that particle number conservation implies:

$$J^\mu{}_{;\mu} = 0\,.$$

If nothing depends upon time, show that this integrates to

$$nu^r |g'|^{1/2} = constant\,,$$

where g' is the determinant of $g_{\mu\nu}$ divided by $\sin^2\theta$ and U^r is... well, you tell me what U^r is.

Solution:

The Schwarzschild metric is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right)c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,. \tag{9}$$

We are operating under the assumption that the particle number is conserved (for relativistic theories, this has to be justified). The 4-current $J^\mu = nU^\mu = (n\gamma(v)c, n\gamma(v)\mathbf{v})$ is conserved. In flat space, this means $\partial_\mu J^\mu = 0$, and then $N = \frac{1}{c}\int d^3x J^0 = const$. In the non-relativistic limit, this is just $\partial n/\partial t + div(n\mathbf{v}) = 0$.

In curved space, this is upgraded to $\nabla_\mu J^\mu = 0$. Such an "upgrade" is unique. Indeed, if we had another, different, tensorial upgrade of $\partial_\mu J^\mu$ to curved space, let's call it S^ν_μ, then the tensor $D^\nu_\mu \equiv S^\nu_\mu - \nabla_\mu J^\mu$ would not be zero. However, in the local rest frame, we must have $S^\nu_\mu \to \partial_\mu J^\mu$ and $\nabla_\mu J^\mu \to \partial_\mu J^\mu$, so in that frame $D^\nu_\mu = 0$. Since D^ν_μ is a tensor, it is then zero in any frame, contrary to the assumption.

A useful formula for a covariant divergence of a vector is (can you prove it?)

$$\nabla_\mu V^\mu = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}V^\mu\right)\,.$$

For the Schwarzschild metric, $\sqrt{|g|} = r^2 \sin\theta$. We have no time dependence and are considering the radial flow only (i.e. $U^\theta = 0$ and $U^\phi = 0$), so $\partial_r \left(r^2 \sin\theta U^r n\right) = 0$, where $U^r = dr/d\tau$. Thus, we have

$$nr^2 U^r = const\,.$$

2b) We move on to energy conservation, $T^{t\nu}{}_{;\nu} = 0$. *(Refer to §4.6 in the notes.) Show that the only nonvanishing affine connection that we need to use is*

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{\partial \ln|g_{tt}|}{\partial r}$$

Derive and solve the energy equation. Show that its solution may be written

$$\left(P + \rho c^2\right) U^r U_t |g'|^{1/2} = constant\,,$$

where $U_t = g_{t\mu}U^\mu$, *and* ρ *is the total energy density of the fluid in the rest frame, including any thermal energy.*

Solution:

The equation $T^{t\nu}{}_{;\nu} = 0$ is explicitly written as

$$T^{t\nu}{}_{;\nu} = \frac{1}{\sqrt{|g|}}\partial_\mu\left(\sqrt{|g|}T^{\mu t}\right) + \Gamma^t_{\mu\lambda}T^{\mu\lambda}\,.$$

The only non-vanishing connection coefficients with the upper index t are

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{1}{2}\frac{\partial \ln|g_{tt}|}{\partial r}\,.$$

(This can be seen immediately from the Euler-Lagrange method of deriving the connection coefficients, discussed in Problem Set 1.) We have (neglecting viscous and higher derivative terms)

$$T^{\mu\nu} = Pg^{\mu\nu} + \left(\rho + P/c^2\right) U^\mu U^\nu$$

and thus

$$T^{\mu t} = Pg^{\mu t} + \left(\rho + P/c^2\right) U^\mu U^t\,.$$

Explicitly,

$$\nabla_\mu T^{\mu t} = g^{tt}\partial_t P + \frac{1}{\sqrt{|g|}}\partial_\mu\left[\sqrt{|g|}\left(\rho + P/c^2\right)U^\mu U^t\right] + 2\Gamma^t_{rt}T^{rt} = 0\,.$$

With $\sqrt{|g|} = r^2\sin\theta$, for a stationary radial flow (where only ∂_r matters) we get

$$\frac{1}{r^2}\partial_r\left[r^2\left(\rho + P/c^2\right)U^r U^t\right] + 2\Gamma^t_{rt}\left(\rho + P/c^2\right)U^r U^t = 0\,,$$

where $\Gamma^t_{rt} = g'_{tt}/2g_{tt}$ (here the prime denotes the derivative w.r.t. r). Then

$$\partial_r \left[r^2 \left(\rho + P/c^2 \right) U^r U^t \right] g_{tt} + r^2 g'_{tt} \left(\rho + P/c^2 \right) U^r U^t = 0 \,,$$

i.e.

$$\partial_r \left[r^2 \left(\rho + P/c^2 \right) U^r U_t \right] = 0 \,,$$

which implies

$$r^2 \left(\rho + P/c^2 \right) U^r U_t = const \,.$$

We remember that $g = r^4 \sin^2\theta$, $g' = g/\sin^2\theta = r^4$, so the above line can be written as

$$|g'|^{1/2} \left(\rho + P/c^2 \right) U^r U_t = const \,.$$

2c) We next define

$$\bar{\omega} = \mu n \,,$$

where μ is the rest mass per particle and $\bar{\omega}$ is a Newtonian density. This is not to be confused with ρ, the true relativistic energy density divided by c^2. Now, P and $\bar{\omega}$ are assumed to be related by a simple power law relationship,

$$P = K\bar{\omega}^\gamma \,,$$

where K is a constant and γ is called the adiabatic index. This is not an entirely artificial problem: it is valid for cold classical particles ($\gamma = 5/3$) or hot relativistic particles ($\gamma = 4/3$). The first law of thermodynamics then tells us that the thermal energy per unit volume is

$$\epsilon = \frac{P}{\gamma - 1}$$

(You needn't derive that here, just use it!) Show that this implies:

$$\rho = \bar{\omega} + \frac{P}{c^2(\gamma - 1)} \,.$$

Solution:

We note that since $\mu = m/N$ is the mass per particle,

$$\bar{\omega} = \mu n = \frac{\mu N}{V} = \frac{m}{V}$$

is the mass density. The total relativistic energy of one free particle is $E = \mu c^2 + K$, where K is kinetic energy. For N non-interacting particles we have $\rho c^2 = NE/V = n\mu c^2 + \epsilon$, where $\epsilon = nK$ (one can add interaction energy, if necessary). So,

$$\rho = \bar{\omega} + \epsilon/c^2$$

or

$$\rho = \bar{\omega} + \frac{P}{c^2(\gamma - 1)} \, .$$

2d) Verify that $g' = r^4$ and using $g_{\mu\nu}U^\mu U^\nu = -c^2$, show that

$$U_t = \left[c^2 - \frac{2GM}{r} + (U^r)^2 \right]^{1/2} \, .$$

(Take care to distinguish U^t and U_t.)

Solution:

For the radial flow with the only non-zero componnets U^t and U^r, the equation $g_{\mu\nu}U^\mu U^\nu = -c^2$ reads

$$g_{tt}(U^t)^2 + g_{rr}(U^r)^2 = -c^2$$

and we find

$$(U^t)^2 = -g^{tt}\left(c^2 + g_{rr}(U^r)^2\right) \, .$$

lowering the index t, we get

$$U_t^2 = -g_{tt}\left(c^2 + g_{rr}(U^r)^2\right) = -g_{tt}c^2 - g_{tt}g_{rr}(U^r)^2 = \left(1 - \frac{2GM}{c^2 r}\right)c^2 + (U^r)^2 \, ,$$

so indeed

$$U_t = \left[c^2 - \frac{2GM}{c^2 r} + (U^r)^2 \right]^{1/2} \, .$$

2e) With $a^2 = \gamma P/\bar{\omega}$ (this is the speed of sound in a nonrelativistic gas), combine our mass and energy conservation equations to show that

$$\left(\mathbf{c^2} + \frac{\mathbf{a^2}}{\gamma - \mathbf{1}}\right)^{\mathbf{2}} \left(\mathbf{c^2} + \mathbf{U^2} - \frac{\mathbf{2GM}}{\mathbf{r}}\right) = \mathbf{const}. \tag{10}$$

We have dropped the superscript r on U^r for greater clarity. How does a^2 depend upon $\bar{\omega}$? The other equation we shall use is just that of mass conservation itself. Show that this may be written as

$$4\pi\bar{\omega}r^2 U = \dot{m} \, ,$$

which defines the net, constant mass accretion rate $\dot{m} < 0$. With a^2 depending entirely on $\bar{\omega}$, and $\bar{\omega} = \dot{m}/(4\pi r^2 U)$, the equation in boldface becomes a single algebraic equation for U as a function of r, and the formal solution to our problem.

Solution:

First, recall that $a^2 \equiv v_s^2 = \partial P/\partial\bar{\omega} = \gamma K \bar{\omega}^{\gamma-1} = \gamma P/\bar{\omega}$. In particular, we see that $a^2 \sim \bar{\omega}^{\gamma-1}$.

The two conservation equations ($\nabla_\mu T^{\mu\nu} = 0$ and $\nabla_\mu J^\mu = 0$) give

$$nU^r r^2 = const\,, \tag{11}$$

$$\left(P + \rho c^2\right) U^r U_t r^2 = const\,, \tag{12}$$

and combining them we find

$$\left(P + \rho c^2\right) \frac{U_t}{n} = const\,.$$

Since $\bar{\omega} = \mu n$, we obtain

$$\frac{\left(P + \rho c^2\right)}{\bar{\omega}} U_t = const\,.$$

However, we found earlier that

$$\rho = \bar{\omega} + \frac{P}{c^2(\gamma-1)}\,.$$

Therefore,

$$P + \rho c^2 = \bar{\omega} c^2 + \frac{\gamma}{\gamma-1} P\,,$$

and, substituting the expression for U_t, we get

$$\left(c^2 + \frac{\gamma}{\gamma-1}\frac{P}{\bar{\omega}}\right)^2 \left(c^2 + (U^r)^2 - \frac{2GM}{r}\right) = const\,,$$

which, remembering that $a^2 = \gamma P/\bar{\omega}$, is the equation we wanted to derive.

We have $nr^2 U^r = const$, so $4\pi r^2 \mu n U^r = const$, or $4\pi r^2 \bar{\omega} U^r = const$. Since $U^r = dr/d\tau$, the constant is the radial mass accretion rate, $\dot{m}$.

2f) Three final simple tasks for now:

i) Show that the constant on the right of the bold equation of problem (2e) is

$$c^2 \left(c^2 + \frac{a_\infty^2}{\gamma-1}\right)^2\,,$$

where a_∞ is the sound speed at infinite distance from the black hole, if the gas starts accreting from rest.

Solution:

Taking the limit $r \to \infty$ and setting $U^r = 0$ at $r \to \infty$ (the gas starts accreting from rest), we find the constant on the right hand side of eq. (10)

$$\left(c^2 + \frac{a_\infty^2}{\gamma - 1}\right)^2 c^2 = const.$$

ii) Show that the Newtonian limit of the equation is

$$\frac{v^2}{2} + \frac{a^2}{\gamma - 1} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1},$$

where v is the ordinary velocity, not the 4-velocity. This is a statement that a quantity known as enthapy (energy plus the work done by pressure) is conserved. This is the original nonrelativistic Bondi 1952 solution for accretion onto a star.

Solution:

We can write

$$\left(1 + \frac{\gamma}{\gamma - 1}\frac{P}{\bar{\omega}c^2}\right)\left(1 + (U^r)^2/c^2 - \frac{2GM}{c^2 r}\right)^{1/2} = \left(1 + \frac{a_\infty^2}{c^2(\gamma - 1)}\right),$$

and then expand for "$c \to \infty$":

$$\left(1 + \frac{\gamma}{\gamma - 1}\frac{P}{\bar{\omega}c^2}\right)\left(1 + (U^r)^2/2c^2 - \frac{GM}{c^2 r} + \cdots\right) = \left(1 + \frac{a_\infty^2}{c^2(\gamma - 1)}\right).$$

To leading order, this gives

$$\frac{\gamma}{\gamma - 1}\frac{P}{\bar{\omega}} + v^2/2 - \frac{GM}{r} = \frac{a_\infty^2}{(\gamma - 1)},$$

where $v = dr/dt$. This can be written as (Bondi, 1952)

$$\frac{a^2}{\gamma - 1} + \frac{v^2}{2} - \frac{GM}{r} = \frac{a_\infty^2}{\gamma - 1}.$$

iii) Show that as r approaches the Schwarzschild radius $R_S = 2GM/c2$, then if $a \ll c$ everywhere, then dr/dt satisfies the condition of a "null geodesic," a fancy way to say the inflow follows the equation of light:

$$\frac{dr}{dt} = -c\left(1 - \frac{R_S}{r}\right).$$

Like stalled photons, from the point of view of a distant observer, the flow never crosses R_S.

Solution:

With $a/c \ll 1$ and $r \to R_S = 2GM/c^2$, the equation

$$\left(1 + \frac{\gamma}{\gamma - 1}\frac{P}{\bar{\omega}c^2}\right)\left(1 + (U^r)^2/c^2 - \frac{2GM}{c^2 r}\right)^{1/2} = \left(1 + \frac{a_\infty^2}{c^2(\gamma - 1)}\right),$$

gives $U^r = \pm c$ (since the gas is infalling, we choose $U^r = -c$). Thus, $dr/d\tau = -c$. Using the Schwarzschild metric, we have for the radial motion

$$ds^2 = -c^2 d\tau^2 = -dr^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 \,. \tag{13}$$

This leads to

$$\left[1 + \left(1 - \frac{2GM}{c^2 r}\right)\right] dr^2 = \left(1 - \frac{2GM}{c^2 r}\right)^2 c^2 dt^2$$

i.e. (in the limit $r \to R_S = 2GM/c^2$)

$$dr^2 = \left(1 - \frac{2GM}{c^2 r}\right)^2 c^2 dt^2$$

which is the photon's geodesic $ds^2 = 0$ in the Schwarzschild metric.

Problem 3

3a) Kinematic and gravitational redshifts. One of the most important observational black hole diagnostics is a calculation of the radiation spectrum from the surrounding disc. In particular we are interested in how the frequency of a photon is shifted due to space-time distortions and relativistic kinematics. Show that:

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R)V^\mu(R)}{p_\mu(E)V^\mu(E)} \, ,$$

where R denotes the received the photon and E the emitted photon, ν is a frequency (not an index here!), p_μ a covariant photon 4-momentum, and V^μ is the normalised 4-velocity in the form $(dt/d\tau, d\mathbf{x}/cd\tau)$ for the emitted material (E) or the distant observer at rest (R).

Solution:

The Schwarzschild metric describing the space-time of the black hole for $r > R_s = 2GM/c^2$ is

$$ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, . \tag{14}$$

The matter (whose backreaction on the metric we neglect) is moving in this space-time along the geodesics, the simplest of which is the circular orbit (considered in Problem 5). This matter emits photons in all directions, and the photons are detected at spatial infinity $r \to \infty$. Note that in this limit the Schwarzschild metric reduces to

$$ds^2 = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \, , \tag{15}$$

which is the Minkowski metric in spherical coordinates. For an observer (or "receiver" R) at $r \to \infty$ therefore, his/her proper time is $\tau = t$ (so the physical meaning of t in Eq. (14) is "the proper time of an observer at spatial infinity"). The 4-coordinate of such an observer in his/her own frame is $X^\mu_R = (ct, \mathbf{0})$ and the 4-velocity is $U^\mu(R) = (c, \mathbf{0})$. We can introduce $V^\mu(R) = U^\mu(R)/c = (1, \mathbf{0})$.

A photon has a 4-momentum $p^\mu = (E/c, \mathbf{p})$, where $E = h\nu$, and $p^\mu p_\mu = 0$. A useful way to compute physical quantities in curved space-time is to form invariants whose ingredients characterise the point of observation and the observed quantity of interest. Here we are interested in the frequency of a photon ν emitted by an element of matter having 4-velocity V^μ. A natural invariant in this case is $p^\mu V_\mu$.

For a photon observed in the reference frame of the receiver, one can form an invariant $p^\mu(R)V_\mu(R)$. In the frame of the receiver, this invariant is

$$p^\mu(R)V_\mu(R) = -\frac{h\nu_R}{c} \tag{16}$$

which gives us an operational way to define ν_R. Note that $p^\mu(R)V_\mu(R)$ can be computed in any convenient frame.

Similarly, if $X^\mu_E = (c\tau_E, 0)$ is the 4-coordinate and $V^\mu(E) = (c, \mathbf{0})$ the (normalised) 4-velocity of the reference frame of the matter orbiting the black hole and emitting a photon with the 4-momentum $p^\mu(E)$, then one can form another invariant, $p^\mu(E)V_\mu(E)$, which in that frame is

$$p^\mu(E)V_\mu(E) = -\frac{h\nu_E}{c}\,. \tag{17}$$

Again, $p^\mu(E)V_\mu(E)$ can be computed in any convenient frame. For example, in the frame of an observer at infinity (the receiver), we have $X^\mu_E = (ct, x^i)$ and $V^\mu(E) = (dt/d\tau, dx^i/cd\tau)$, where τ is the proper time of the matter element emitting the photon.

We have then

$$\frac{\nu_R}{\nu_E} = \frac{p_\mu(R)V^\mu(R)}{p_\mu(E)V^\mu(E)}\,. \tag{18}$$

3b) In the problem at hand, the observer views the disc edge-on, in the plane of the disc. The gas moves in circular orbits. Show that in t, r, θ, ϕ coordinates for the 0, 1, 2, 3 *components,*

$$V^\mu(R) = (1,0,0,0)\,, \qquad V^0_E = V^\mu_0(1,0,0,d\phi/cdt)\,,$$

with $V^0_E = dt/d\tau$.

Then, using $g_{\mu\nu}V^\mu V^\nu = -1$, conclude that

$$V^0_E = \left(1 - 3GM/rc^2\right)^{-1/2}\,.$$

You may use a result from problem (5c) below. (You will prove it later!)

Solution:

This has been already explained in section 3a. In coordinates t, r, θ, ϕ used by an observer at infinity, his/her own normalised 4-velocity is $V^\mu(R) = (1,0,0,0)$ (only time changes in his/her frame), and the normalised 4-velocity of an emitter is $V^\mu(E) = (dt/d\tau, 0, 0, d\phi/cd\tau)$ (the ordit is circular, so only ϕ changes).

Explicitly, the expression $g_{\mu\nu}V^\mu V^\nu = -1$ reads

$$g_{tt}\left(\frac{dt}{d\tau}\right)^2 + g_{\phi\phi}\left(\frac{d\phi}{cdt}\right)^2\left(\frac{dt}{d\tau}\right)^2 = -1\,.$$

Denoting the angular velocity $\Omega = d\phi/dt$ and using the explicit form of the metric, we find

$$\left(V^0_E\right)^2\left[-\left(1 - \frac{2GM}{c^2 r}\right) + r^2\frac{\Omega^2}{c^2}\right] = -1\,.$$

For circular orbits in Schwarzschild metric, we have (see Problem 5 or other sources)

$$\Omega^2 = \frac{GM}{r^3} .$$

Ths gives

$$V_E^0 = \left(1 - 3GM/rc^2\right)^{-1/2} .$$

3c) Finally, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \left(1 + \frac{\Omega p_\phi(E)}{cp_0(E)}\right)^{-1} .$$

A result of problem 3 from Problem Set 1 may be useful.

From disk material moving at right angles across the line of sight, ν_R/ν_E reduces to

$$\left(1 - 3GM/rc^2\right)^{1/2} .$$

Why?

From disk material moving precisely along the line of sight, show that

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} / \left(1 \pm \left(rc^2/GM - 2\right)^{-1/2}\right) .$$

(Hint: $g^{\nu\rho} p_\nu p_\rho = 0$.) Interpret the $\pm$ sign. In general, the photon paths must be calculated from the dynamical equations to determine the $p(E)$ ratio.

Solution:

Computing the right hand side of Eq. (18) in the frame of the receiver, we have

$$\frac{\nu_R}{\nu_E} = \frac{p_0(R)V^0(R)}{p_0(E)V^0(E) + p_\phi(E)V^\phi(E)} ,$$

where $V^0(R) = 1$, $V^\phi(E) = \Omega V^0(E)/c$, and

$$V_E^0 = \left(1 - 3GM/rc^2\right)^{-1/2} ,$$

as found in section 3b). A result of problem 3 from Problem Set 1 states that $p_0(R)/p_0(E) = 1$, provided $p^\mu = mdx^\mu/d\tau \equiv dx^\mu/d\lambda$ satisfies the geodesic equation - and it does (here $\lambda = \tau/m$ is the affine parameter which can be used for null geodesics as well).

This immediately gives

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} \left(1 + \frac{\Omega p_\phi(E)}{cp_0(E)}\right)^{-1} . \tag{19}$$

The two special cases are shown in Fig. 1:

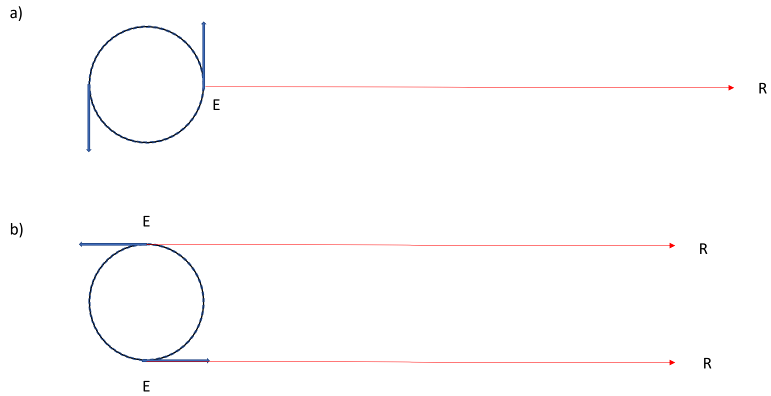

FIG. 1: An illustration to Problem 3c: a) The disk material is moving at right angles across the line of sight; b) The disk material moving precisely along the line of sight.

a) If the disk material is moving at right angles across the line of sight (see part a) in Fig. 1), we have $p_\phi(E) = 0$, since in this case photons are moving radially outward to reach the receiver R (and thus the only non-zero componets of p_μ are p_0 and p_r). Then the formula (20) gives

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} . \tag{20}$$

b) If the disk material is moving precisely along the line of sight (and thus the only non-zero componets of p_μ are p_0 and p_ϕ), we can find the ratio $p_\phi(E)/p_0(E)$ from the condition reflecting the masslessness of the photon: $p^2 = g_{\mu\nu}p^\mu p^\nu = g^{\mu\nu}p_\mu p_\nu = 0$. Explicitly, $g^{00}(p_0)^2 + g^{\phi\phi}(p_\phi)^2 = 0$, or

$$\Omega\frac{p_\phi(E)}{p_0(E)} = \pm\sqrt{-\frac{g^{00}}{g^{\phi\phi}}}\frac{\Omega}{c} = \pm\frac{1}{\sqrt{rc^2/GM - 2}} . \tag{21}$$

Here the $\pm$ sign is determined by the relative sign of the components $p_\phi = g_{\phi\phi}d\phi/d\lambda$ and $p_0 = g_{00}cdt/d\lambda$ of the photon's 4-momentum. When the direction of photon's propagation is opposite to the direction of rotating matter V^ϕ (as shown in the upper part of the b) section in Fig. 1), we have $p_\phi < 0$. Since $p_0 < 0$ as well (note that $g_{00} < 0$), the relative sign of the two components of momentum is positive, and one has to chose $''+''$ sign in Eq. (21). Similarly, when the photon is emitted along the direction of rotation of matter, one chooses $''-''$ sign in Eq. (21). Then

$$\frac{\nu_R}{\nu_E} = \left(1 - \frac{3GM}{rc^2}\right)^{1/2} / \left(1 \pm \left(rc^2/GM - 2\right)^{-1/2}\right) ,$$

where the $''+''$ sign corresponds to the photon emitted by the matter receding from the observer at infinity (see the upper part of the b) section in Fig. 1) and the $''-''$ sign corresponds to the photon emitted by the matter moving towards the observer at infinity (see the lower part of the b) section in Fig. 1). In the first case, we have $\nu_R < \nu_E$ (redshift), in the second - $\nu_R > \nu_E$ (blueshift).

Of course, these are special cases of photon's motion. In general, the photon's geodesic equation should be used to determine the ratio $p_\phi(E)/p_0(E)$.

Problem 4

4a) The perihelion advance of Mercury. In the notes we found that the differential equation for $u = 1/r$ *for Mercury's orbit could be written as* $u = u_N + \delta u$ *with the Newtonian solution* u_N *given by*

$$u_N = \frac{GM}{J^2}\left(1 + \epsilon\cos\phi\right)$$

and the differential equation for δu *is*

$$\frac{d^2\delta u}{d\phi^2} + \delta u = \frac{3(GM)^3}{c^2 J^4}\left(1 + 2\epsilon\cos\phi + \epsilon^2\cos^2\phi\right)\ . \tag{22}$$

Show that this is equivalent to solving the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a\left(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2\right)\ ,$$

where $a = 3(GM)^3/c^2J^4$ *and* $b = 1 + \epsilon^2$*.*

To solve this, try a solution of the form

$$\delta u = A_0 + A_1\phi e^{i\phi} + A_2\phi e^{2i\phi}\ ,$$

where the A's are constants. Why do we need an additional factor of ϕ *in the* A_1 *term?*

Solution:

Eq. (22) is an inhomogeneous ODE of the type

$$Ly(x) = F(x)\ ,$$

where $L = d^2/dx^2 + 1$ is a second-order linear differential operator. A solution can be written in the form

$$y(x) = y_0(x) - \int G(x - x')F(x')dx'\ ,$$

where y_0 is a solution of the homogeneous equation, $Ly_0 = 0$, and the Green's function $G(x - x')$ obeys the equation

$$LG(x - x') = -\delta(x - x')\ .$$

The Green's function can be found either explicitly or using the Fourier transform. Here we do not follow this standard method but rather use an ansatz for the solution, as the problem suggests. However, you are strongly advised to contruct the solution this way as an additional exercise.

The source function on the right hand side of Eq. (22) contains terms whose frequencies are multiple integers of the fundamental eigenfrequency $\omega_0 = 1$ of the homogeneous equation. This leads to resonance phenomena and the term linear in ϕ in the solution (can you show that this is a generic feature - e.g. by using the Green's function approach?).

Since $\cos^2\phi = (1+\cos 2\phi)/2$, the ODE (22) is equivalent to the real part of the equation

$$\frac{d^2\delta u}{d\phi^2} + \delta u = a\left(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2\right) , \tag{23}$$

where $a = 3(GM)^3/c^2J^4$ and $b = 1+\epsilon^2$, since

$$\mathrm{Re}\left[a\left(b + 2\epsilon e^{i\phi} + \epsilon^2 e^{2i\phi}/2\right)\right] = a\left(b + 2\epsilon\cos\phi + \frac{\epsilon^2}{2}\cos 2\phi\right) .$$

4b) Show that the solution for $u = u_N + \delta u$ is

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi + \epsilon a\phi\sin\phi\,.$$

Since a is very small, show that this equivalent to

$$u = \frac{GM}{J^2}\left[1 + \epsilon\cos(\phi(1-\alpha))\right] + ab - \frac{a\epsilon^2}{6}\cos 2\phi\,.$$

Solution:

Using the ansatz

$$\delta u = A_0 + A_1\phi e^{i\phi} + A_2\phi e^{2i\phi} ,$$

computing the derivatives and comparing with Eq. (23), we identify

$$A_0 = ab\,, \qquad A_1 = -ia\epsilon\,, \qquad A_2 = -a\epsilon^2/6\,.$$

Therefore,

$$\delta u = ab - ia\epsilon\phi e^{i\phi} - \frac{a\epsilon^2}{6}\phi e^{2i\phi} ,$$

whose real part is

$$\delta u = ab + a\epsilon\phi\sin\phi - \frac{a\epsilon^2}{6}\cos 2\phi$$

or

$$\delta u = \frac{3(GM)^3}{c^2J^4}\left[1 + \frac{\epsilon^2}{2} + \epsilon\phi\sin\phi - \frac{\epsilon^2}{6}\cos 2\phi\right] .$$

The full solution is

$$u = u_N + \delta u = \frac{GM}{J^2}\left(1 + \epsilon\cos\phi\right) + \frac{3(GM)^3}{c^2 J^4}\left[1 + \frac{\epsilon^2}{2} + \epsilon\phi\sin\phi - \frac{\epsilon^2}{6}\cos 2\phi\right] \tag{24}$$

or

$$u = \frac{GM}{J^2} + ab - \frac{a\epsilon^2}{6}\cos 2\phi + \frac{GM}{J^2}\epsilon\cos\phi + \epsilon a\phi\sin\phi\,. \tag{25}$$

To linear order in $\alpha = aJ^2/GM \ll 1$, Eq. (25) is equivalent to the equation

$$u = \frac{GM}{J^2}\left[1 + \epsilon\cos(\phi(1-\alpha))\right] + ab - \frac{a\epsilon^2}{6}\cos 2\phi\,. \tag{26}$$

Indeed, expanding for small $\alpha\phi$, we find

$$\begin{aligned}&\frac{GM}{J^2}\epsilon\cos\left(\phi - \alpha\phi\right) = \frac{GM}{J^2}\epsilon\cos\phi + \frac{GM}{J^2}\epsilon\alpha\phi\sin\phi + O(\alpha^2)\\ &= \frac{GM}{J^2}\epsilon\cos\phi + \frac{3(GM)^3}{J^4c^2}\epsilon\phi\sin\phi + O(\alpha^2)\,.\end{aligned}$$

4c) In the equation for u, the first two terms in a cause tiny (and unmeasurable) distortions in the shape of the ellipse, but do not affect the 2π perodicity in ϕ of the orbit. Show however that the final term, proportional to GM/J^2, results in a periastron advance of

$$\delta\phi = 6\pi\left(\frac{GM}{cJ}\right)^2$$

each orbit. This is the classic Einstein result.

Solution:

Periastron is the point nearest to a star in the path of a body orbiting that star. The term

$$\frac{GM}{J^2}\epsilon\cos\left(\phi - \alpha\phi\right)$$

has the period in ϕ of

$$T = \frac{2\pi}{1-\alpha} \approx 2\pi(1+\alpha)\,.$$

(Only the term linear in $\alpha \ll 1$ is relevant since the expression (26) is only valid to linear order in α.)

Therefore, the periastron's advance in each orbit is

$$\delta\phi = 6\pi\left(\frac{GM}{cJ}\right)^2\,.$$

Problem 5

5a) Black hole orbits. In Newtonian theory, the energy equation for a test particle in orbit around a point mass is

$$\frac{v^2}{2} + \frac{l^2}{2r^2} - \frac{GM}{r} = \mathcal{E}\,,$$

where r is radius, v is the radial velocity, l the angular momentum per unit mass, $\mathcal{E}$ the constant energy per unit mass, and $-GM/r$ is of course the potential energy. For the Schwarzschild solution show that the integrated geodesic equation may also be written in the form

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathcal{E}_S\,,$$

where r is the standard radial coordinate, l_S and $\mathcal{E}_S$ are constants, $\Phi_S(r)$ is an effective potential function, and $v_S = dr/d\tau$. Determine l_S and $\mathcal{E}_S$ in terms of the fundamental angular momentum and energy constants J and E from lecture (or the notes). Express $\Phi_S(r)$ in terms of l_S, $\mathcal{E}_S$, the speed of light c, GM and r. The form of l_S, $\mathcal{E}_S$, and $\Phi_S(r)$ should be chosen to go over to their Newtonian counterparts in the limit $E \to c^2$, $c \to \infty$ $E - c^2 \to$ finite.

Solution:

For massive objects moving along the geodesics parametrised by the proper time of the moving object, $x^\mu = x^\mu(\tau)$, we can, since $ds^2 = -c^2 d\tau^2 = g_{\mu\nu}dx^\mu dx^\nu$, write

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2\,, \tag{27}$$

where $\dot{x}^\mu \equiv dx^\mu/d\tau$. Of course, we can use any other affine parameter $\lambda = \alpha\tau + \beta$, then

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2/\alpha^2\,,$$

where $\dot{x}^\mu \equiv dx^\mu/d\lambda$. Recall that an affine parameter is the parameter of the 4-dimensional geodesic $x^\mu = x^\mu(\lambda)$ such that the geodesic's equation of motion has the form

$$\ddot{x}^\mu + \Gamma^\mu_{\lambda\sigma}\dot{x}^\lambda \dot{x}^\sigma = 0\,. \tag{28}$$

If the parameter is diffrerent from the affine one, the equation of motion has extra terms (which is fine in principle but less convenient).

For massless particles, we have $ds^2 = 0$. Their geodesics still obey the equation (28), where λ is not the proper time. In the massless case,

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = 0\,.$$

In both cases, the geodesic equation follows from the Euler-Lagrange equations applied to $\mathcal{L}$:

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \,.$$

If $\mathcal{L}$ is independent of a certain coordinate x^μ, we have a symmetry and the corresponding Noether conserved quantity

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = 0 \qquad \Rightarrow \qquad \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = const \,.$$

For example, the Schwarzschild metric is independent of t and ϕ, and therefore we have 2 integrals of motion, $\partial\mathcal{L}/\partial t = const$ and $\partial\mathcal{L}/\partial\phi = const$. Here the constants on the right hand side are related to the initial conditions. From these 2 equations, we can express $\dot{t}$ and $\dot{\phi}$ via the integration constants (initial conditions) and substitute into Eq. (27). There is yet another conserved quantity (recall that the Schwarzschild metric is static and spherically symmetric, and spherical symmetry implies the existence of *two* conserved quantities - L_z and L^2 in terms of angular momentum): $\theta = \pi/2 = const$. This can be seen directly from the equation of motion satisfied by θ. As a result, we have only one dynamical variable in Eq. (27), namely, $\dot{r}$.

We shall follow notations in the Lecture notes. Eq. (27) becomes Eq. 338 there:

$$\dot{r}^2 + Bc^2\left(1 + \frac{J^2}{Er^2}\right) = \frac{c^4}{E} \,,$$

where $\dot{r} = dr/d\tau$, $B = 1 - 2GM/c^2r$, and E and J are the Noether integration constants, as discussed above (their physical meaning can be understood by considering $r \to \infty$ limit). With $v_s \equiv \dot{r}$, we have

$$\frac{v_s^2}{2} + \frac{J^2c^2}{2Er^2} - \frac{GM}{r}\left(1 + \frac{J^2}{Er^2}\right) = \frac{c^2}{2}\left(\frac{c^2}{E} - 1\right) .$$

We now identify $l_S^2 = J^2c^2/E$ as well as

$$\mathcal{E}_S = \frac{c^2}{2}\left(\frac{c^2}{E} - 1\right) \,, \qquad \Phi_S(r) = -\frac{GM}{r}\left(1 + \frac{J^2}{Er^2}\right)$$

to get

$$\frac{v_S^2}{2} + \frac{l_S^2}{2r^2} + \Phi_S(r) = \mathcal{E}_S \,.$$

5b) Sketch the effective potential $l_S^2/2r^2 + \Phi_S(r)$. Prove that there is always a potential minimum in Newtonian theory, but that this is not the case in general relativity. What is the mathematical condition for the existence of a potential minimum for $\Phi_S(r)$, and what does it mean physically if it does not exist?

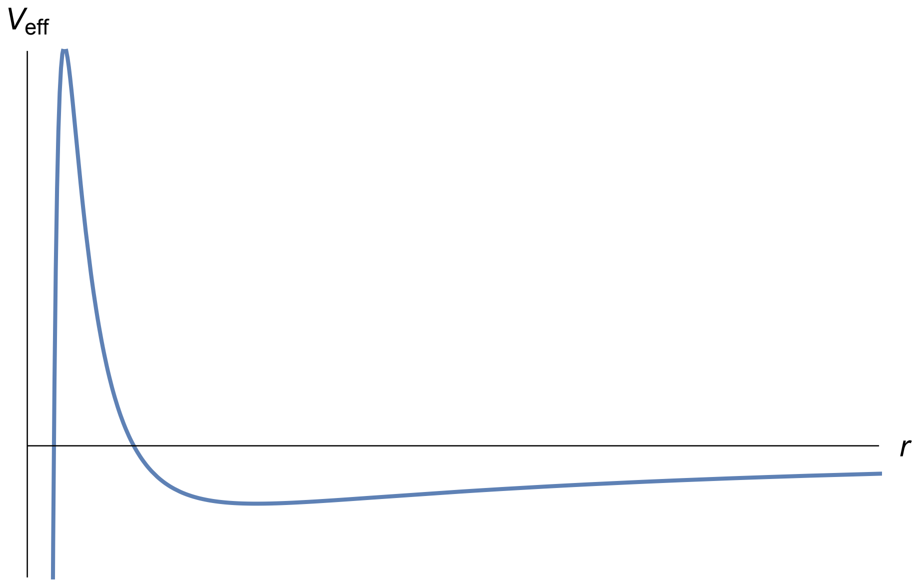

FIG. 2: An illustration to Problem 5b: The effective potential $V_{eff} = l_S^2/2r^2 + \Phi_S(r)$ as a function of r. In the Newtonian limit, the term proportional to $-1/r^3$ in $\Phi_S(r)$ vanishes and so at small r the potential is dominated by the positive term $\sim 1/r^2$.

Solution:

The effective potential is shown schematically in Fig. 2. At small r, the dominant contribution to the potential comes from the $1/r^3$ term which is negative. This term is absent in the Newtonian limit and in that case the potential increases indefinitely at small r as $1/r^2$.

By analysing $V'_{eff}(r) = 0$, one can see that there are no extrema for

$$E > \frac{1}{12}\left(\frac{Jc^2}{GM}\right)^2 .$$

5c) Show that for the Schwarzschild metric, circular orbits satisfy

$$\Omega^2 = \frac{GM}{r^3} ,$$

exactly the Newtonian form. Here $\Omega = d\phi/dt$ at the coordinate location r, where dt is the proper time interval at infinity. Derive expressions for E and J in terms of GM, c^2 and r.

Solution:

The equation of motion for $r(\lambda)$ is (see Lecture Notes):

$$A\dot{r}^2 + \frac{J^2}{r^2} - \frac{c^2}{B} = -E .$$

Taking the derivative w.r.t. λ and taking $r = const$ gives:

$$-\frac{2J^2}{r^3} + \frac{B'c^2}{B^2} = 0 .$$

We also have

$$\dot{\varphi}^2 = J^2/r^4 = \left(\frac{d\varphi}{dt}\right)^2 \frac{1}{B^2} = \frac{\Omega^2}{B^2}\,.$$

Therefore,

$$\Omega^2 = \frac{c^2 B'}{2r} = \frac{GM}{r^3}\,.$$

Then

$$J = \frac{r^2\Omega}{B} = \frac{\sqrt{GMr}}{1-2GM/c^2r}\,.$$

Also,

$$B = \frac{1}{E}\left(c^2 - \frac{BJ^2}{r^2}\right) = \frac{1}{E}\left(c^2 - \frac{r^2\Omega^2}{B}\right)\,.$$

Thus,

$$E/c^2 = \frac{1}{B^2}\left(B - \frac{r^2\Omega^2}{c^2}\right) = \frac{1-3GM/c^2r}{(1-2GM/c^2r)^2}\,.$$

5d) Below what value of r does $\Phi_S(r)$ not have any local extrema? (Answer: $6GM/c^2$.)

Solution:

From $V'_{eff}(r) = 0$ with $E = c^4J^2/12G^2M^2$ (critical case) we have

$$\frac{c^2}{B^2}\left(1 - \frac{3GM}{c^2r}\right) = \frac{c^4r^4\Omega^2}{G^2M^2B^2}\,.$$

With $\Omega^2 = GM/r^3$ we have

$$1 - \frac{3GM}{c^2r} = \frac{c^2r}{12GM}\,,$$

i.e. $r = 6GM/c^2 = 3R_s$.