THE SIGN OF FOUR: COORDINATES, 4-VECTORS and 4-TENSORS

"That's its business," said Lambert. "If Balbus says it's the same bulk, why, it's the same bulk, you know."

"Well, I don't believe it," said Hugh.

"You needn't," said Lambert. "Besides, it's dinner-time. Come along." They found Balbus waiting dinner for them, and to him Hugh at once propounded his difficulty.

"Let's get you helped first," said Balbus, briskly cutting away at the joint. "You know the old proverb, 'Mutton first, mechanics afterwards'?"

The boys did not know the proverb, but they accepted it in perfect good faith, as they did every piece of information, however startling, that came from so infallible an authority as their tutor. They ate on steadily in silence, and, when dinner was over, Hugh set out the usual array of pens, ink, and paper, while Balbus repeated to them the problem he had prepared for their afternoon's task.

— Excerpt from *A Tangled Tale*, by Lewis Carroll

NOTE: Problem 5 is optional.

1.) Consider the following thought:

"Special relativity holds for frames moving at constant relative velocity, but of course acceleration requires general relativity because the frames are noninertial."

Wait, what? Nonsense! Special relativity is a formulation encompassing all nongravitational physics, and it certainly doesn't collapse before simple kinematical accleration. On the other hand, acceleration, even just uniform accleration in one dimension, is not without its connections to general relativity. We shall explore some of them here. *For ease of notation, let us set $c = 1$.* In part (d) we'll put c back.

1a.) Let us first ask what we mean by "uniform acceleration." After all, a rocket approaching the speed of light c can't change its velocity at a uniform rate forever without exceeding c at some point. Go into the frame moving instantaneously at velocity v, *with* the rocket relative to the "lab." By definition, in this frame, the instantaneous rocket velocity v' is zero. Now, wait a time dt' later, as measured in this frame. The rocket will now have a velocity dv' in this same frame. What we mean by constant acceleration is that $dv'/dt' \equiv a'$ is constant. By contrast, the acceleration measured in the fixed lab is certainly not constant! The question is, how is the lab acceleration $a = dv/dt$ related to the truly constant a'?

To answer this, let $V = v/\sqrt{1-v^2}$, the spatial part of the 4-vector V^α associated with the ordinary velocity $\boldsymbol{v}$, and the same for V' and v'. Assume, for the moment, that the primed and unprimed frames differ by some arbitrary velocity w. The 4-velocity differentials are then given by:

$$dV' = (dV - w\,dV^0)/\sqrt{1-w^2}$$

where $V^0 = 1/\sqrt{1-v^2}$. Explain.

1b.) Now, set $w = v$. We thereby go into the frame in which $v' = 0$; the rocket is instantaneously at rest. Prove that $dv = dv'(1 - v^2)$. (Remember, v and v' are the *ordinary* velocities.) From here, prove that

$$\frac{dv}{dt} = a'(1 - v^2)^{3/2}.$$

1c.) Show that, starting from rest at $t = t' = 0$,

$$v = \frac{a't}{\sqrt{1 + a'^2 t^2}}, \qquad a't = \sinh(a't'),$$

and hence show that (for $x = 0$ at $t = t' = 0$):

$$v = \tanh(a't'), \qquad x = \frac{1}{a'}[\cosh(a't') - 1]$$

The integrals are not difficult; do them yourselves.

1d.) Let's use these results to construct a full coordinate transformation from the lab frame x, t to the accelerating x', t' frame. A good start is to guess a transform of the form

$$t = A(x')\sinh(a't') + B(x'), \qquad x = A(x')\cosh(a't') + C(x')$$

where A, B, and C depend only upon x'. Then on $x' =$ constant surfaces, $dx/dt = \tanh(a't') = v$, which is indeed what we need.

By definition, constant t' surfaces are constant-time surfaces in the (x', t') frame that move instantaneously with velocity $v = \tanh(a't')$ with respect to the (x, t) frame. On such a surface, $dt'/dx' = 0$. We fix the origin by demanding that as $t' \to 0$, $x \to x'$. We fix our clock by demanding that as $t' \to 0$, $t \to t'$ at the rocket location $x' = 0$. (This must be done locally: since A depends on x', this time agreement can be exact at only one value of x'.) Show that these constraints force B and C to be constant, and that B in particular must vanish.

Finally, put the speed of light c back into the equations, demand that x goes to x' at $t' = 0$, and show that

$$ct = \left(\frac{c^2}{a'} + x'\right)\sinh(a't'/c), \qquad x = \left(\frac{c^2}{a'} + x'\right)\cosh(a't'/c) - \frac{c^2}{a'}$$

1e.) Show that the invariant Minkowski line element may be written in x', t' coordinates as:

$$c^2 d\tau^2 = c^2 dt^2 - dx^2 = \left(1 + \frac{a'x'}{c^2}\right)^2 c^2 dt'^2 - dx'^2.$$

Provide a physical interpretation of your result in terms of a gravitational redshift. How do you interpret the region $x' \leq -c^2/a'$? (Review the results of 1d.)

2.) *Orbits with relativistic kinematics.* The relativistic equations for a test mass with ordinary velocity $\boldsymbol{v}$ in a potential of the form $-\alpha/R$, where α is a constant, are given by:

$$\gamma c^2 - \frac{\alpha}{R} = E$$

$$\gamma R^2 \dot{\phi} = J$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$ and we use standard R, ϕ cylindrical coordinates in the orbital plane. $\dot{\phi}$ is an ordinary time derivative, i.e. the time is measured in the fixed "lab" frame, and E and J are constants. These equations are in fact the exact classical relativistic equations for an electron in orbit about a positive point charge, but we can solve them as a poor man's substitute for proper general relativity by setting $\alpha = GM$. (To the best of my knowledge, no one even tried this during the years between special and general relativity, when the distinction was less clear.) Later in the course, we can compare these "ersatz orbits" with those of the true theory. In any case, you'll have solved exactly the correct, classical relativistic atomic orbit problem!

2a.) Give a physical interpretation to each of these equations. What do the constants E and J represent physically? Be precise.

2b.) Show that

$$\left(\frac{dR}{d\phi}\right)^2 = \left(\frac{\dot{R}}{\dot{\phi}}\right)^2 = \frac{R^4}{J^2c^2}\left(E^2 - c^4 + \frac{2\alpha E}{R} + \frac{1}{R^2}[\alpha^2 - J^2c^2]\right)$$

(You may wish to begin by first solving for $\dot{\phi}$ in terms of R alone.)

2c.) Set $u = 1/R$ and derive a second order linear differential equation for u:

$$\frac{d^2u}{d\phi^2} + \mu^2 u = \frac{\alpha E}{J^2c^2}$$

where

$$\mu^2 = 1 - \frac{\alpha^2}{J^2c^2}$$

2d.) Finally, show that, up to an unimportant orbital phase constant, the most general solution of this equation takes the form

$$u = \frac{\alpha E}{\mu^2 J^2 c^2}[1 + \epsilon\cos(\mu\phi)]$$

where ϵ is an arbitrary constant of integration. Show that when $\alpha \ll Jc$, this orbit corresponds to a near ellipse, but an ellipse in which the point of closest approach advances by an angle

$$\Delta\phi = \frac{\pi\alpha^2}{J^2c^2}$$

every orbit. We will see that general relativity also predicts such an advance of the *perihelion,* as it is called, but by an amount which is six times larger! The precise match of GR with observations of Mercury's orbit was the theory's first great success.

3.) *Recognising tensors.* One way to prove that something is a vector or tensor is to show explicitly that it satisfies the coordinate transformation laws. This can be a long and arduous procedure if the tensor is complicated with many indices. There is another way, usually much better!

Show that if V_ν is an arbitrary covariant vector and the combination $T^{\mu\nu}V_\nu$ is known to be a contravariant vector (note the free index μ), then

$$\left(T'^{\mu\nu} - T^{\lambda\sigma}\frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\sigma}\right)V'_\nu = 0$$

Why does this prove that $T^{\mu\nu}$ is a tensor? Does your proof actually depend on the rank of the tensors involved?

4.) *What about $d^2x_\mu/d\tau^2$?* The geodesic equation in standard form gives us an expresssion for $d^2x^\mu/d\tau^2$ in terms of the affine connection, $\Gamma^\mu_{\nu\lambda}$. For the covariant coordinate x_μ, show that

$$\frac{d^2x_\mu}{d\tau^2} = \frac{1}{2}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\frac{\partial g_{\nu\rho}}{\partial x^\mu}$$

Refer to section 4.7 in the notes if help is needed. Under what conditions is $dx_0/d\tau = V_0 \equiv V_t$ a constant of the motion?

5*.) *What is "the spatial part" of a metric?: An* ***optional*** *problem for further study.* It is easy, even trivial, to get the proper time from a metric. One simply sets all the spatial $dx^i = 0$ in the invariant interval $g_{\mu\nu}dx^\mu dx^\nu$, and reads off a proper time of

$$d\tau = \sqrt{-g_{00}}\,dx^0/c.$$

This is what a local inertial observer reads off on their watch. So to get " the spatial part" of the metric, let's call it dl^2, do we just take whatever is left over from setting $dx^0 = 0$, i.e. $dl^2 = g_{ij}dx^i dx^j$? Not quite; it depends on the application. The point is that one cannot enforce global simultaneity by setting dx^0 everywhere

How does a physical observer actually measure a distance? They take a light ray, bounce it off a mirror some distance dl away, measure their local (proper) time on their watch $d\tau$ for the light to go and come back, and then determine the distance via $dl = cd\tau/2$. Let's go with that.

5a.) Show that for a *diagonal* metric tensor (all $g_{0i} = g_{i0} = 0$), this procedure gives

$$dl^2 = g_{ij}dx^i dx^j,$$

just as we expect. (In interpreting your results, you will find it useful to think of the light ray as leaving at a negative time interval, and returning after a positive time interval.)

5b.) Show that for a general metric tensor $g_{\mu\nu}$, with nonvanishing $g_{0i} = g_{0i}$, this procedure gives

$$dl^2 = \gamma_{ij}dx^i dx^j, \text{ where } \gamma_{ij} = g_{ij} - (g_{0i}g_{0j}/g_{00})$$

The metric tensor of a rotating black hole (the Kerr metric) actually has $g_{0\phi} = g_{\phi 0}$ components, so this formula is very relevant here. We see that the spatial part of the metric may contain mixed time-indexed terms!

5c.) Using the $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$ relations, show that

$$g^{ij}\gamma_{jk} = \delta^i_k$$

and that, defining γ^{ij} by raising indices via $\gamma^{ij} \equiv g^{ik} g^{jm} \gamma_{km}$, leads to

$$\gamma^{ij} = g^{ij},$$

the "pure spatial part" of $g^{\mu\nu}$. γ^{ij} defined this way is indeed the inverse of γ_{ij}. Hence, we are justified in regarding $\gamma_{ij} dx^i dx^j$ as the invariant interval in its own three-dimensional space, with inverse γ^{ij}, within the more encompassing four-dimensional $g_{\mu\nu}$ spacetime. (Note that this also shows that the indices on γ_{ij} may be raised with γ^{ij}.)

5d.) Show that $\det g_{\mu\nu} = g_{00} \det \gamma_{ij}$, which is consistent with identifying γ_{ij} as the spatial metric. You may find it useful to recall that the determinant of a matrix is unchanged when a multiple of one row is added to another.