Lecture 9

Relativistic wave equations

1. Klein-~~Gordon~~ eq. (Schrödinger, 1926)

<u>Non-rel. particles</u>: $E = \vec{P}^2/2m$

$E \to i\hbar \frac{\partial}{\partial t}$, $p_i = -i\hbar \frac{\partial}{\partial x_i}$

$$\Rightarrow i\hbar \frac{\partial \psi}{\partial t} = \hat{H} \psi$$

This eq. is parabolic ($\partial/\partial t$) and is not Lorentz-covariant.

Parabolic eqs (Navier-Stokes, diffusion etc) $\Rightarrow$ infinite signal propagation speed.

e.g. diffusion eq. $\frac{\partial \phi}{\partial t} = D \nabla^2 \phi$

solution $\phi(t, \vec{r}) = \frac{1}{(4\pi D t)^{n/2}} e^{-\frac{r^2}{4Dt}}$

$\phi(0, \vec{r}) = \delta(\vec{r})$.

At <u>any</u> $t > 0$, solution is nonzero at $r = r_*$ for $\forall r_* \Rightarrow$ propagates with inf. speed.

Hyperbolic eq. → finite propagation speed.

E.g. Maxwell's eqs:

$$\Box A^\mu = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) A^\mu = 0$$

But Maxwell's eq. are classical, not quantum. Need $\hbar$.

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$x^\mu = (ct, x, y, z)$$

$$x_\mu = \eta_{\mu\nu} x^\nu = (ct, -x, -y, -z),$$

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}.$$

$$\frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{c\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \partial_\mu .$$

$$\frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \partial^\mu .$$

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 :$$ Lor.-invar.

$$\Box A^\mu = 0 \quad \text{photon}$$

$$A^\mu \sim e^{ikx}, \quad k^\mu = \left(\frac{\hbar\omega}{c}, \hbar\vec{k}\right)$$

$$k^\mu k_\mu = 0 \quad (\text{massless})$$

Massive rel. particle: $p^\mu = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$

$$\mathcal{E}^2/c^2 - \vec{p}^{\,2} = m^2c^2$$

$$\mathcal{E} \to i\hbar\frac{\partial}{\partial t}, \quad p_i = -i\hbar\frac{\partial}{\partial x_i}$$

$$\Rightarrow \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x_i^2} + \frac{m^2c^2}{\hbar^2}\right)\Phi(t,\vec{x}) = 0$$

$$(\Box + m^2)\Phi(x) = 0$$

<u>Note</u>: $(\Box - m^2)\Phi(x) = 0$ in $(-+++)$ metric, where $\Box = -\frac{1}{c^2}\partial^2/\partial t^2 + \vec{\nabla}^2$.

<u>Note</u>: rescale $x = L\bar{x}$, $[\bar{x}] = 1$.

Then $\left(\bar{\Box} + \left(\frac{L}{\lambda\!\!\!^{-}_c}\right)^2\right)\Phi = 0$, $\lambda\!\!\!^{-}_c = \frac{\hbar}{mc}$ is Compton wave-length.

For $L \gg \lambdabar_c$, $\Phi \approx 0 \Rightarrow$
non-triv. solutions for $L \sim \lambdabar_c$.

In the non-rel. case: $\rho = |\psi|^2$ prob. density

$$\frac{\partial}{\partial t}|\psi|^2 = \frac{\partial \psi^*}{\partial t}\psi + \psi^* \frac{\partial \psi}{\partial t}$$

Use $i\hbar \frac{\partial \psi}{\partial t} = H\psi$, $\quad -i\hbar \frac{\partial \psi^*}{\partial t} = H^* \psi^*$

$$\Rightarrow \frac{\partial}{\partial t}|\psi|^2 = -\frac{i\hbar}{2m}\left(\psi \nabla^2 \psi^* - \psi^* \nabla^2 \psi\right) =$$

$$= -\frac{i\hbar}{2m}\partial_i\left(\psi \partial_i \psi^* - \psi^* \partial_i \psi\right) = -\operatorname{div}\vec{j},$$

$$j_i = \frac{i\hbar}{2m}\left(\psi \partial_i \psi^* - \psi^* \partial_i \psi\right).$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \operatorname{div}\vec{j} = 0$$

$$\Rightarrow \frac{\partial w}{\partial t} + \oint_\Sigma \vec{j}\cdot d\vec{\Sigma} = 0, \quad w = \int d^3x\, |\psi|^2.$$

Try the same for KG:

$$\psi^*\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi - \psi\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi^* = 0$$

$\Rightarrow$

$$\frac{\partial}{\partial t}\left[\frac{i\hbar}{2mc^2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right)\right] + \operatorname{div}\frac{\hbar}{2im}\left[\psi^*\nabla\psi - \psi\nabla\psi^*\right] = 0$$

Let

$$\rho_{KG} = \frac{i\hbar}{2mc^2}\left(\psi^*\partial_t\psi - \psi\partial_t\psi^*\right) = \frac{\hbar}{mc^2}\operatorname{Im}\psi\,\partial_t\psi^*$$

ρ_{KG} is real but can be positive or negative: KG is second order eq. in time deriv. $\Rightarrow$ $\psi(0)$ and $\partial_t\psi(0)$ must be specified independently as initial conditions.

<u>First-order eq.</u> $\Rightarrow$ state vectors evolve by unitary transf., e.g. $\psi(t,\vec{x}) = e^{\frac{i\hat{H}t}{\hbar}}\psi(0,\vec{x})$ $= \hat{U}(t)\psi(0,\vec{x})$, $U^+U = 1$.

But then Lor. inv. implies the eq. must be also first order in spatial derivatives

$\Rightarrow$ Dirac eq.

Solutions of KG eq:

1) free solutions:

$$\psi^{\pm} = e^{\pm i(\vec{k}\cdot\vec{r} - \omega t)}$$

$$\omega = \sqrt{\vec{k}^2 c^2 + m^2 c^4/\hbar^2} > 0$$

ψ^+: particle with $\varepsilon = \hbar\omega$ and $\vec{p} = \hbar\vec{k}$.

ψ^-: particle with $\varepsilon = -\hbar\omega < 0$?

$\vec{p} = -\hbar\vec{k}$

Introduce $\psi_c(\vec{r},t) = \hat{C}\psi(\vec{r},t)$,

$\hat{C}$: charge conjugation operator: $\hat{C}\psi = \psi^*$.

KG eq. is invar. under $\hat{C}$ $\Rightarrow$ ψ and ψ_c obey the same eq. (particle and antiparticle).

2) External <u>scalar</u> potential: eg weak grav field 9-7

$$\left[-c^2\hbar^2\vec{\nabla}^2 + m^2c^4 + 2mc^2 U(\vec{r},t)\right]\psi =$$

$$= -\hbar^2\frac{\partial^2}{\partial t^2}\psi$$

<u>Non-rel. limit:</u> $\varepsilon = mc^2 + \frac{p^2}{2m} + U(\vec{r},t) + \dots$

Stationary states: $\psi \sim e^{-i\varepsilon t/\hbar} \to$ formally Schrödinger eq. with $E \equiv \frac{\varepsilon^2 - m^2c^4}{2mc^2}$.

$\Rightarrow$ can find bound states with $E < 0$ in the usual way. Note that for some values of parameters can have $\varepsilon_0^2 = 0 \Rightarrow$ and $\varepsilon^2 < 0$.

$\Rightarrow$ instability related to spontaneous creation of particles/antiparticles (they obey the same eq) $\Rightarrow$ single-particle eq. is not applicable.

Full gravit. field:

$$\frac{1}{\sqrt{|g|}}\partial_\mu\left[g^{\mu\nu}\sqrt{|g|}\,\partial_\nu\phi\right] + \left(\frac{mc}{\hbar}\right)^2\phi = 0$$

Now suppose the particle has non-zero charge e. How to introduce electromagnetic interaction?

„<u>Minimal coupling</u>" vs „<u>Non-minimal coupling</u>"

Ultimately, dictated by agreement with experiment. We know:

$$\dot{\vec{p}} = \underbrace{e\vec{E} + \frac{e}{c}[\vec{v}\times\vec{B}]} + \dots (?)$$

this can be reproduced from

$$\mathcal{L} = -mc^2\sqrt{1-\frac{\vec{v}^2}{c^2}} + \frac{e}{c}\vec{A}\cdot\vec{v} - e\Phi \quad \text{or}$$

$$\mathcal{H} = \sqrt{m^2c^4 + c^2\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2} + e\Phi. \Rightarrow$$

<u>Minimal coupling</u>: $\vec{p} \to \vec{p} - \frac{e}{c}\vec{A}$,

$E \to E - e\Phi$

If $(?)$ is detected $\Rightarrow$ $\mathcal{H}$ should be modified $\Rightarrow$ non-min coupling terms as in higher-der. gravity.

3) charged scalar particles: interaction with external electromagnetic field

<u>Minimal coupling</u>: $\begin{cases} \vec{p} \to \vec{p} - \frac{e}{c}\vec{A} \\ E \to E - e\phi \end{cases}$

$$\left(i\hbar\frac{\partial}{\partial t} - e\phi\right)^2 \psi = \left(\frac{\hbar c}{i}\vec{\nabla} - e\vec{A}\right)^2 \psi + m^2c^4\psi .$$

Hydrogen atom: $e\phi = -Ze^2/r$, $\vec{A} = 0$.

$\Rightarrow$

$$E = mc^2\left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{\ell + 1/2} - \frac{3}{4}\right) + \cdots\right]$$

$\alpha \equiv \frac{e^2}{4\pi\hbar c} \simeq 1/137$

(fine structure constant)

<u>incorrect</u> (disagrees with experiment).

(fine structure term)

$\downarrow$

SPIN $\mapsto$ Dirac eq.