Problem: find $S_\ell(k)$ from $U(r)$ by solving

$$\varphi''(r) + \left[k^2 - \frac{\lambda^2 - 1/4}{r^2} - U(r)\right]\varphi = 0 .$$

Here $R(r) = \varphi/r$, $\lambda \equiv \ell + 1/2$.

Investigate singularity structure of $S_\ell(k)$ for various classes of $U(r)$. Develop approximate methods (Born series etc).

Need to analyze the ODE.

Assume $\lim_{r\to 0} r^2 U(r) = 0$ (can be generalized).

Then two local solutions at $r = 0$ are

$$\begin{cases} \varphi_1 = r^{\lambda + 1/2} + \dots \\ \varphi_2 = r^{-\lambda + 1/2} + \dots \end{cases} \quad (r \to 0) .$$

or $\varphi(\lambda, k, r)$ and $\varphi(-\lambda, k, r)$.

At $r \to \infty$ assume $U(r) \to 0$:

local solutions are $f(\lambda, \pm k, r)$, where

$$\lim_{r\to\infty} e^{ikr} f(\lambda, k, r) = 1, \quad \text{i.e. } f \sim e^{-ikr}$$

is the incoming wave. The solution $f(\lambda, k, r)$ is known as Jost solution.

For example, for $U = 0$ the Jost sol. is

$$f_0(\lambda, k, r) = \sqrt{\frac{\pi k r}{2}}\, e^{-\frac{i\pi}{2}(\lambda + 1/2)} H_\lambda^{(2)}(kr),$$

where $H_\nu^{(2)}(z)$ is the Hankel function.

One can prove existence and investigate analyticity properties with resp. to k, λ of $f(\lambda, \pm k, r)$ and $\varphi(\pm\lambda, k, r)$ by writing integral eqs. for them

(see De Alfaro, Regge „Potential Scattering",
North-Holland Publ, 1965).

To solve ODE, need connection formulas:

$$\varphi(\lambda,k,r) = \underline{\underline{A(\lambda,k)}}\, f(\lambda,k,r) + \underline{\underline{B}}\, f(\lambda,-k,r)$$

Compute Wronskian

$$Wr[f,\varphi] = f'\varphi - f\varphi' =$$

$$= f'\left(Af + Bf(-k)\right) - f\left(Af' + Bf'(-k)\right) =$$

$$= B\left[f'(k)f(-k) - f(k)f'(-k)\right] =$$

$$= B\, Wr\left[f(k), f(-k)\right] = B \cdot 2ik$$

$$\Rightarrow B = \frac{Wr\left[f(k),\varphi(k)\right]}{2ik} \equiv \frac{f(\lambda,k)}{2ik},$$

where $f(\lambda,k)$ is the <u>Jost function</u>.

Similarly, $A = -\dfrac{f(\lambda,-k)}{2ik}$

$\Rightarrow$

$$\varphi(\lambda,k,r) = \frac{f(\lambda,k)}{2ik} f(\lambda,-k,r) -$$

$$- \frac{f(\lambda,-k)}{2ik} f(\lambda,k,r)$$

$$\Rightarrow \boxed{S_\ell(k) = \frac{f(\lambda,k)}{f(\lambda,-k)} e^{i\pi(\lambda - 1/2)}}$$

One can show that $f(\lambda,-k) = f^*(\lambda,k^*)$, $\lambda \in \mathbb{R}$. Zeros of $f(\lambda,-k)$ and poles of $f(\lambda,k)$ determine singularities of $S_\ell(k)$

$\Rightarrow$ bound states, virtual states, resonances as in 1-dim. case.

The Born approximation

For stationary states with $E>0$, need to solve

$$\left(-\frac{\hbar^2}{2m}\Delta + U(\vec{r})\right)\psi(\vec{r}) = E\psi(\vec{r})$$

or $$(\Delta + k^2)\psi = \frac{2mU}{\hbar^2}\psi,$$

where $k^2 = 2mE/\hbar^2$.

Solution: $\psi(\vec{r})$ obeys the integral eq:

$$\psi^{\pm}(\vec{r}) = \psi_0^{\pm}(\vec{r}) + \int G^{\pm}(\vec{r},\vec{r}\,')\,\frac{2mU(\vec{r}\,')}{\hbar^2}\,\psi^{\pm}(\vec{r}\,')\,d^3x',$$

where $G^{\pm}(\vec{r},\vec{r}\,')$ is the Green's function of the Helmholtz eq. in 3d:

$$(\Delta + k^2)\,G^{\pm}(\vec{r},\vec{r}\,') = \delta^{(3)}(\vec{r}-\vec{r}\,'),$$

$$G^{\pm}(\vec{r},\vec{r}\,') = -\frac{e^{\pm ik|\vec{r}-\vec{r}\,'|}}{4\pi|\vec{r}-\vec{r}\,'|}.$$

exercise (optional): find $G^{\pm}$ using Fourier transform.

<u>exercise (opt)</u>: find G for the Helmholtz operator $\Delta + k^2$ in n dimensions.

<u>Note</u>: 1) $\psi_0^\pm(\vec{r})$ is the solution of the homogeneous eq. $(\Delta + k^2)\,\psi_0^\pm = 0$, i.e.

$$\psi_0^\pm(\vec{r}) = e^{\pm i\vec{k}\vec{r}}.$$

2) G^+ corresp. to outgoing <u>spherical</u> wave at $r \to \infty$, G^- – to incoming one.

Scattering problem: $\psi^+(\vec{r}) \equiv \psi(\vec{r})$ obeys

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}} - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\, U(\vec{r}\,')\,\psi(\vec{r}\,')\,d^3x'$$

At $r = |\vec{r}| \to \infty$:

$$|\vec{r}-\vec{r}\,'|^2 = (x-x')^2 + (y-y')^2 + (z-z')^2 =$$

$$= x^2\left(1 - \frac{2x'}{x} + \ldots\right) + y^2\left(1 - \frac{2y'}{y} + \ldots\right) + z^2\left(1 - \frac{2z'}{z} + \ldots\right) =$$

$$= x^2 + y^2 + z^2 - 2xx' - 2yy' - 2zz' + \ldots =$$

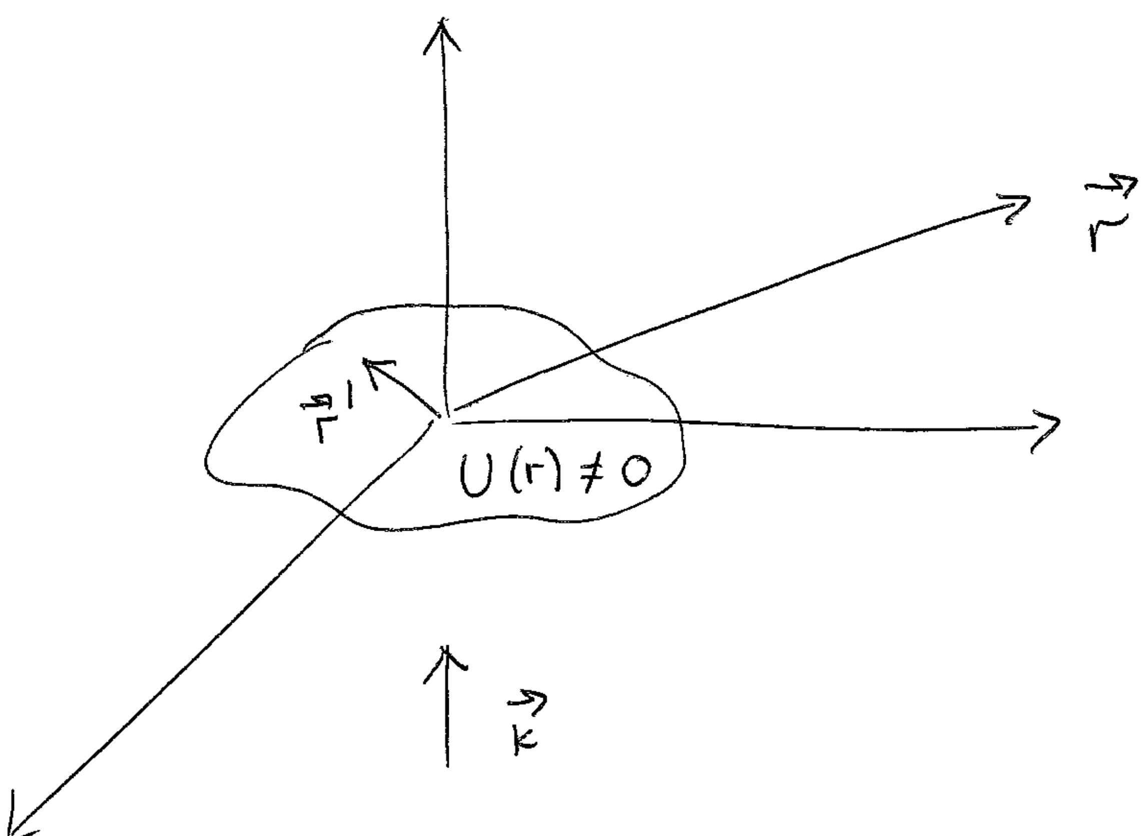
$$= |\vec{r}|^2 \left(1 - \frac{2\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^2} + \dots\right)$$

$$\Rightarrow |\vec{r} - \vec{r}\,'| = |\vec{r}| \left(1 - \frac{\vec{r}\cdot\vec{r}\,'}{|\vec{r}|^2} + \dots\right)$$

Then at $r \to \infty$:

$$\psi(\vec{r}) = e^{i\vec{k}\vec{r}} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k}\,'\cdot\vec{r}\,'} U(\vec{r}\,')\psi(\vec{r}\,')d^3x' ,$$

where $\vec{k}\,' \equiv k\frac{\vec{r}}{r}$



Perturbative solution (Born series):

if $U(\vec{r})$ contains a small parameter λ, expand

$$\psi(\vec{r}) = \psi_0(\vec{r}) + \lambda\psi_1(\vec{r}) + \dots,$$

$$\psi_0(\vec{r}) = e^{i\vec{k}\cdot\vec{r}}.$$

Then up to $O(\lambda^2)$ we have at $r\to\infty$:

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{i(\vec{k}-\vec{k}')\vec{r}'} U(\vec{r}')\, d^3x'$$

$\Rightarrow$ the scattering amplitude in the (first) Born approximation:

$$f^{(1)}(\theta) = -\frac{m}{2\pi\hbar^2}\int e^{i\vec{q}\cdot\vec{r}'} U(\vec{r}')\, d^3x',$$

where $\vec{q} \equiv \vec{k} - \vec{k}'$, recall $\vec{k}' \equiv k\frac{\vec{r}}{r}$.

Note: $\vec{q}^{\,2} = (\vec{k}-\vec{k}')^2 = \vec{k}^2 + \vec{k}'^2 - 2\vec{k}\cdot\vec{k}' =$

$$= 2k^2 - 2k^2\cos\theta = 2k^2(1-\cos\theta) = 4k^2\sin^2\frac{\theta}{2}$$

$\Rightarrow q = |\vec{q}| = 2k\sin\theta/2$, where θ is the angle between $\vec{k}$ and $\vec{k}'$.

For $U(\vec{r}) = U(r)$ can integrate over the angles:

$$f^{(1)}(\theta) = -\frac{m}{2\pi\hbar^2} \cdot 2\pi \int_0^\infty dr' \int_0^\pi d\theta' \, r'^2 \sin\theta' \cdot e^{iqr'\cos\theta'} U(r')$$

With $t = \cos\theta'$ get

$$f^{(1)}(\theta) = -\frac{2m}{\hbar^2} \int_0^\infty dr' \, \frac{r' \sin qr'}{q} \, U(r') \, .$$

<u>Properties</u> of Born approximation for $f(\theta)$:

1) $d\sigma/d\Omega$ is a function of q only $\Rightarrow$ it depends on $E \sim k^2$ and θ in combination $k^2(1-\cos\theta)$.

2) $f^{(1)}(\theta)$ is real (i.e. optical theorem is violated).

3) $d\sigma/d\Omega$ is indep. of the sign of V.

4) for slow particles $f^{(1)} \simeq -\frac{2m}{\hbar^2} \int_0^\infty r^2 U(r) dr$,

i.e. indep. of θ (this is in agreement with our previous result).

<u>example</u>: a Yukawa pot. $U(r) = \frac{U_0 e^{-\mu r}}{r}$

$$f^{(1)}(\theta) = -\frac{2mU_0}{\hbar^2} \frac{1}{q^2+\mu^2}, \quad \text{where}$$

$q^2 = 4k^2 \sin^2 \theta/2$.

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2mU_0}{\hbar^2}\right)^2 \frac{1}{\left(4k^2\sin^2\theta/2 + \mu^2\right)^2}.$$

Note: with $\mu \to 0$, $U_0 = Z_1 Z_2 e^2$, $\vec{p} = \hbar\vec{k}$, $E = \vec{p}^{\,2}/2m$ we get:

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 e^2}{4E}\right)^2 \frac{1}{\sin^4\theta/2}, \quad \text{i.e. the}$$

Rutherford scattering cross-section (!).

<u>Remark</u>: cross-section for the Coulomb

pot. $U(r) = -\alpha/r$ quantum scattering can be obtained exactly by solving the Schrödinger eq. in terms of the degenerate hypergeometric function ${}_1F_1(a; b; z)$ (see e.g. L-L, vol. III, § 135). The Born approximation in this case is <u>exact</u>.

Moreover, this exact result coincides with the classical scattering formula (!!!) — the only known potential with this property.

<u>Validity of the Born approximation</u>

I. <u>Necessary conditions</u>

$$\psi^{(1)}(\vec{r}) = -\frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} U(\vec{r}\,') e^{i\vec{k}\cdot\vec{r}\,'} d^3x'$$

should not change $\psi^{(0)}(\vec{r})$ too much.

Expect strong influence of $U(\vec{r})$ around $\vec{r}=0$.

Since $\psi_0 = e^{i\vec{k}\vec{r}} \simeq 1$ near $\vec{r}=0$,
the condition is (for $U(\vec{r}) = U(r)$):

$$\frac{m}{2\pi\hbar^2}\left| \int \frac{e^{ikr'}}{r'} U(r') e^{i\vec{k}\cdot\vec{r}\,'} d^3x' \right| \ll 1 .$$

a) <u>slow particles</u> $(k\bar{a} \ll 1)$

$$\frac{m}{2\pi\hbar^2} 4\pi \left| \int_0^\infty r U(r) dr \right| \ll 1$$

$$\frac{2m\bar{a}^2}{\hbar^2}\bar{U} \ll 1 \quad \text{or } \bar{K} = \frac{\hbar^2}{2m\bar{a}^2} \gg \bar{U},$$

where $\bar{U}\bar{a}^2 \equiv \left| \int_0^\infty r U(r) dr \right|$. $B \equiv \frac{\bar{U}}{\bar{K}} \ll 1$
(Born parameter)

b) <u>fast particles</u>

The condition above becomes after integration over angles:

$$\frac{m}{k\hbar^2} \left| \int_0^\infty dr\, U(r) \left(e^{2ikr} - 1\right) \right| \ll 1$$

For $kr \gg 1$ the exponent is oscillating rapidly $\Rightarrow$ small contribution to the integral.

$$\frac{m\bar{a}^2}{\hbar^2 k\bar{a}}\,\bar{U} \ll 1\,, \qquad \text{where}$$

$$\bar{U}\bar{a} \equiv \left|\int_0^\infty U(r)\,dr\right|.$$

i.e. $B = \dfrac{\bar{U}}{\bar{K}} \ll k\bar{a}$, $\quad \underline{k\bar{a} \gg 1}$.

Generally, the Born approx. is quite good at higher energies.

II. <u>Sufficient condition</u> : convergence of the Born series.