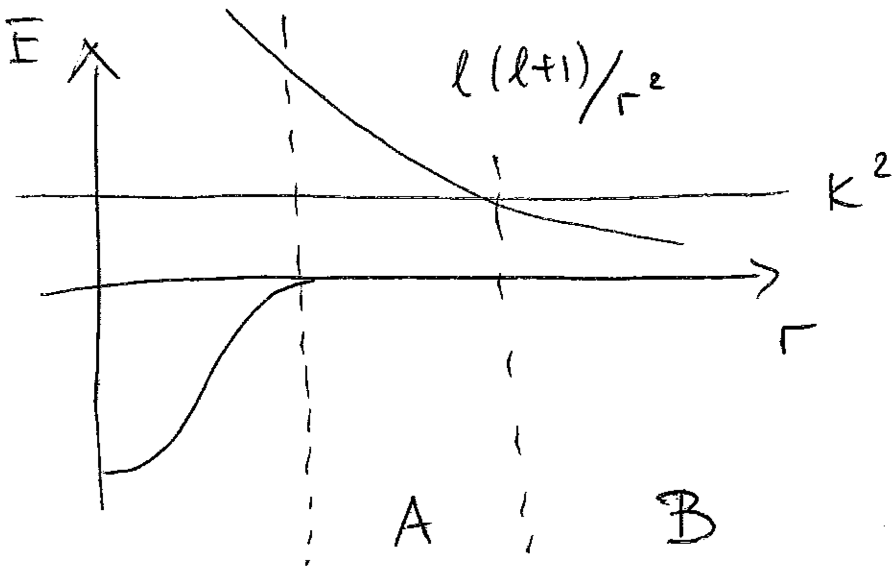
S1

Scattering of slow particles by a short-range potential

/ Lecture 7 supplem. material /

$$\varphi'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right]\varphi = 0$$



Let $x = kr$.

$$\varphi''_{xx} + \left[1 - \frac{\ell(\ell+1)}{x^2} - \frac{U(x/k)}{k^2}\right]\varphi = 0$$

In A: $\quad \varphi'' - \frac{\ell(\ell+1)}{x^2}\varphi = 0$

$$\varphi_A = C_1 x^{\ell+1} + C_2 x^{-\ell}$$

since $\varphi = x^\alpha \Rightarrow \alpha(\alpha-1) - \ell(\ell+1) = 0 \Rightarrow \begin{matrix}\alpha = -\ell \\ \alpha = \ell+1\end{matrix}$

In B: $\varphi'' + \left(1 - \frac{\ell(\ell+1)}{x^2}\right)\varphi = 0$

$$\varphi_B = A x^{1/2} J_{\ell+1/2}(x) + B x^{1/2} N_{\ell+1/2}(x)$$

Take $x \to 0$:

$$\varphi_B \simeq \frac{A}{\Gamma\left(\ell+\frac{1}{2}+1\right) 2^{\ell+\frac{1}{2}}} x^{\ell+1} + \frac{B\, 2^{\ell+\frac{1}{2}}}{\Gamma\left(-\ell+\frac{1}{2}\right)} x^{-\ell},$$

where $\Gamma(z)$ is the Gamma-function.

This matches with $\varphi_A(x)$ if

$$\frac{B}{A} = \frac{c_2}{c_1} \frac{\Gamma\left(-\ell+\frac{1}{2}\right)}{2^{2\ell+1}\Gamma\left(\ell+\frac{1}{2}+1\right)} =$$

$$= \frac{c_2}{c_1} \frac{(-1)^\ell}{(2\ell+1)\left[(2\ell-1)!!\right]^2}.$$

For $x \to \infty$

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \pi/4\right)$$

$$N_\nu(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\nu\pi}{2} - \pi/4\right)$$

Collecting coefficients at e^{ix} and e^{-ix}, we find

$$S_\ell = e^{i2\delta_\ell} = \frac{1 - iB/A}{1 + iB/A} .$$

$$|B/A| \ll 1 \Rightarrow S_\ell \approx 1 - \frac{2iB}{A} .$$

$$\Rightarrow f(\theta) = -\frac{1}{k} \sum_{\ell=0}^{\infty} \frac{c_2}{c_1} \frac{(-1)^\ell}{[(2\ell-1)!!]^2} P_\ell(\cos\theta)$$

The ratio c_2/c_1 is determined by U at small x, e.g. $\varphi(\bar{a}) = 0$ for $U \to \infty$ at $r = \bar{a}$ $(x = k\bar{a})$. $\Rightarrow$ $-\frac{c_2}{c_1} = (k\bar{a})^{2\ell+1}$.

$$\Rightarrow f(\theta) \approx \bar{a} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} P_{\ell}(\cos\theta)}{[(2\ell-1)!!]^2} (k\bar{a})^{2\ell}$$

$$(2\ell-1)!! = 1 \cdot 3 \cdot 5 \cdot \cdots$$

$$(-1)!! = 1$$

For $k\bar{a} \ll 1 \Rightarrow f(\theta) \approx \bar{a} + O((k\bar{a})^2)$

$$\Rightarrow \sigma_{el} = 4\pi\bar{a}^2 \left[1 + O\left((k\bar{a})^2\right) \right].$$

Moral : for short-range potentials, partial scattering amplitudes with $l > 0$ are suppressed for slow particles.