Lecture 7

Exact solution for the wave f. of a <u>free</u> particle with momentum $\vec{p} = \hbar\vec{k}$ moving in the positive z direction:

$$\psi_0 = e^{ikz} = \sum_{l=0}^{\infty} (2l+1)\, i^l j_l(kr) P_l(\cos\theta)$$

For $U(r) \neq 0$,

$$\psi = \sum_{l=0}^{\infty} (2l+1)\, i^l R_l(k,r) P_l(\cos\theta).$$

We know that at $r \to \infty$

$$j_l(kr) \sim \frac{1}{2ikr}\left(e^{i(kr - l\pi/2)} - e^{-i(kr - l\pi/2)}\right)$$

$\Rightarrow$ at $r \to \infty$ $R_l(k,r)$ should have the form

$$R_l(k,r) \sim \frac{1}{2ikr}\left(S_l(k)\, e^{i(kr - l\pi/2)} - e^{-i(kr - l\pi/2)}\right).$$

<u>Problem:</u> find $S_l(k)$ for a given $U(r)$.

Inverse problem: find $U(r)$ from $S_\ell(k)$. Need $S_0(k)$ but also asymptotics of the bound states wave f. in $U(r)$, if there are any. Then $U(r)$ can be reconstructed by solving the Gelfand-Levitan-Marchenko integral eq.

The solution $\psi(\vec{r})$ can be written as

$$\psi(\vec{r}) = e^{ikz} + \psi_{scatter}(\vec{r}), \quad \text{where}$$

$$\psi_{scat}(\vec{r}) \sim \frac{1}{2ik} \frac{e^{ikr}}{r} \sum_{\ell=0}^{\infty} (2\ell+1)(S_\ell(k)-1) P_\ell(\cos\theta)$$

for $r \to \infty$, i.e. $\psi(\vec{r})$ is asymptotically

$$\psi(\vec{r}) \sim e^{ikz} + \frac{e^{ikr}}{r} f(\theta), \quad \text{where}$$

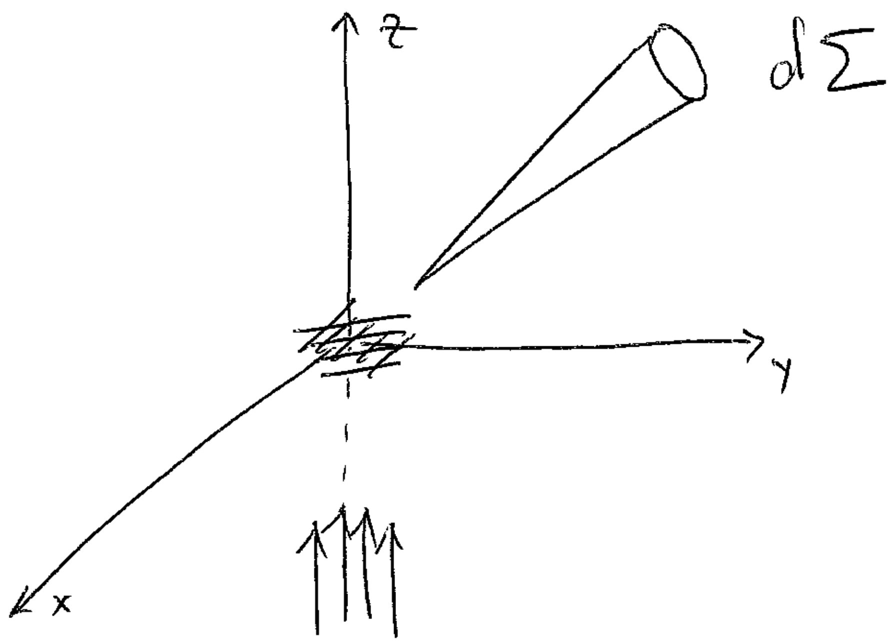
$$f(\theta) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1)(S_\ell - 1) P_\ell(\cos\theta) \quad \text{is}$$

the scattering amplitude

(H. Faxen, J. Holtsmark, 1927), (Lord Rayleigh, 1894).

<u>Note</u>: $[f(\theta)] = L$ (length)

Differential cross-section



<u>Recall</u>: $\dfrac{\partial |\psi|^2}{\partial t} + \operatorname{div} \vec{j} = 0$,

$$\vec{j} = \frac{\hbar}{2im}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$ (e.g. LL, vol III, §19)

For $\psi_{inc} = e^{ikz}$; $\vec{j} = (0, 0, j_{inc})$,

$$j_{inc} = \frac{i\hbar}{2m}\left(\psi_{inc}\partial_z \psi^*_{inc} - \psi^*_{inc}\partial_z \psi_{inc}\right) = \frac{\hbar k}{m} = v.$$

Probability for a scattered particle to pass through $d\Sigma = r^2 d\Omega$ at $r \to \infty$ per unit time is (elastic scatt. $v_{in} = v_{out}$):

$$j_{scatt} = v \frac{|f|^2}{r^2} d\Sigma = v |f|^2 d\Omega .$$

$$d\sigma_{el} = \frac{j_{scatt}}{j_{inc}} = |f(\theta)|^2 d\Omega \quad \text{is called}$$

the differential cross-section:

$$d\sigma_{el} = 2\pi \sin\theta \, |f(\theta)|^2 d\theta .$$

Note: $[\sigma] = L^2$

Note: Tacitly assumed no interference between e^{ikz} and $\frac{e^{ikr}}{r} f(\theta)$ at large r. This is true for $\theta \neq 0$ but the issue is subtle, see e.g. K. Gottfried, Quantum Mechanics.

$$d\sigma_{el} = |f(\theta)|^2 d\Omega$$

$$|f|^2 = \frac{1}{4k^2} \sum_{\ell_1 \ell_2} (2\ell_1+1)(2\ell_2+1) P_{\ell_1}(\cos\theta) P_{\ell_2}(\cos\theta)$$

$$\times |S_{\ell_1} - 1| \, |S_{\ell_2} - 1|)$$

<u>Note</u> : $\int_0^\pi \sin\theta \, P_k(\cos\theta) P_\ell(\cos\theta) \, d\theta = \frac{2\delta_{k\ell}}{2\ell+1}$

$$\Rightarrow \sigma_{el} = 2\pi \int_0^\pi \sin\theta \, |f(\theta)|^2 d\theta =$$

$$= \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) |1 - S_\ell|^2 .$$

For <u>elastic</u> scattering $|S_\ell| = 1 \Rightarrow S_\ell = e^{i2\delta_\ell}$

$$\Rightarrow \sigma_{el} = \frac{4\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1) \sin^2 \delta_\ell = \sum_\ell \sigma_\ell \, .$$

In addition to elastic channel, one may

have inelastic channels, where the intensity of the outgoing wave is less than that of the incoming wave (e.g. due to absorption by target). In this case, $|S_\ell| \leq 1$.

$$\sigma_{inel} = \frac{\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1 - |S_\ell|^2\right)$$

$$\sigma_{tot} = \sigma_{el} + \sigma_{inel} = \frac{2\pi}{k^2} \sum_{\ell=0}^{\infty} (2\ell+1)\left(1 - \mathrm{Re}\, S_\ell\right).$$

It will be useful to introduce partial amplitudes by

$$f(\theta) = \sum_{\ell=0}^{\infty} (2\ell+1) f_\ell(k) P_\ell(\cos\theta),$$

i.e. $f_\ell = \frac{1}{2ik}(S_\ell - 1)$.

Note:

$$f(0) = \frac{1}{2ik} \sum_{\ell=1}^{\infty} (2\ell+1)\left(\operatorname{Re} S_\ell + i \operatorname{Im} S_\ell - 1\right)$$

$$\Rightarrow \operatorname{Im} f(0) = \frac{1}{2k} \sum_{\ell=1}^{\infty} (2\ell+1)(1 - \operatorname{Re} S_\ell).$$

Compare with $\sigma_{tot} \Rightarrow$

$$\sigma_{tot} = \frac{4\pi}{k} \operatorname{Im} f(0)$$

(optical theorem). Here $f(\theta)$ is the elastic channel scattering amplitude.

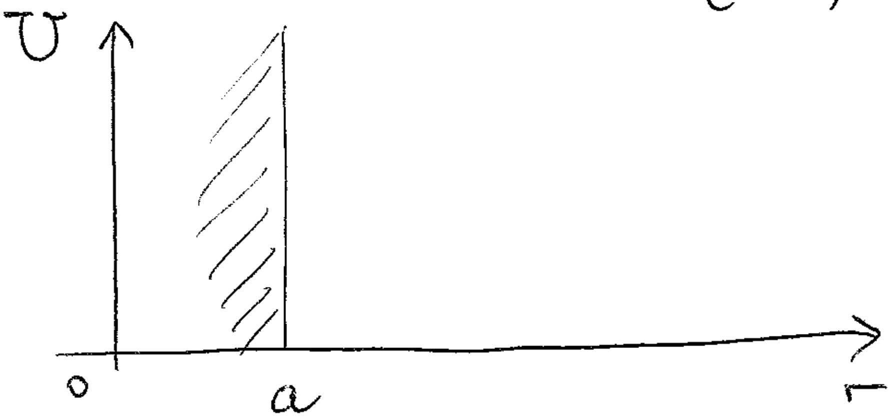
To find $S_\ell(k)$, need to solve radial Schröd. eq.

$$R'' + \frac{2}{r} R' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right] R = 0$$

With $R(r) = \varphi/r$, the eq. is

$$\varphi'' + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right] \varphi = 0.$$

<u>example</u>: $U(r) = \begin{cases} \infty, & r < a, \\ 0, & r > a. \end{cases}$



$$\varphi(a) = 0$$

Consider s-waves ($l=0$) for simplicity:

$$\varphi'' + k^2\varphi = 0 \qquad (r > a)$$

$$\varphi(r) = A \sin kr + B \cos kr$$

$$\varphi(a) = 0 \Rightarrow \varphi(r) = A\left(\sin kr - \tan ka \cos kr\right) =$$

$$= \beta e^{ikr} - \alpha e^{-ikr}$$

$$\beta = A\left(\frac{1}{2i} - \frac{1}{2}\tan ka\right), \quad \alpha = A\left(\frac{1}{2i} + \frac{1}{2}\tan ka\right)$$

$$S_0 = \frac{\beta}{\alpha} = \frac{1 - i\tan ka}{1 + i\tan ka} = e^{-i2ka} = e^{i2\delta_0}$$

$\Rightarrow \delta_0 = -ka$.

$$\sigma_0 = \frac{4\pi}{k^2}\sin^2\delta_0 = \frac{4\pi}{k^2}\sin^2 ka \approx 4\pi a^2$$

for $ka \ll 1$

(slow particles)

<u>Remark</u>: for short-range potentials and slow particles one can show that

$$\sigma \approx \sigma_0 \approx 4\pi\bar{a}^2 \left[1 + O\left((k\bar{a})^2\right)\right],$$

where the <u>scattering length</u> $\bar{a}$ is defined as

$$-\frac{1}{\bar{a}} = \lim_{k\to 0}\left[k\cot\delta_0(k)\right].$$

Slow (long-wavelength) particles do not resolve details of the potential $\Rightarrow$ cross-section is isotropic and essentially constant. See LL, § 132 of Vol. III.