# Lecture 5

Finding the poles of $S(k)$ for $\lambda \ll 1$:

Eq 1:

$$\begin{cases} \sin 2\xi = -2\lambda \xi e^{2\eta}, \\ \cos 2\xi = 2\lambda \eta e^{2\eta} - e^{2\eta}. \end{cases}$$

Let $X = 2\xi$, $Y = 2\eta$:

$$\begin{cases} \sin X = -\lambda X e^{Y}, \\ \cos X = \lambda Y e^{Y} - e^{Y}. \end{cases}$$

$$X = \varepsilon_0 + \lambda \varepsilon_1 + \lambda^2 \varepsilon_2 + \ldots$$

$$Y = \delta_0 + \lambda \delta_1 + \lambda^2 \delta_2 + \ldots$$

Subst. into eqs, expand in $\lambda$. Show that

$$\begin{cases} \varepsilon_0 = n\pi, & n \text{ odd} \\ \delta_0 = 0 \end{cases}$$

$$\begin{cases} \varepsilon_1 = n\pi, & n \text{ odd} \\ \delta_1 = 0 \end{cases}$$

$$\begin{cases} \varepsilon_2 = n\pi, & n \text{ odd} \\ \delta_2 = -\dfrac{n^2\pi^2}{2} \end{cases}$$

$$\Rightarrow \quad \xi_n^+ = \zeta_n^+ + i\eta_n^+ = \frac{n\pi}{2}\left(1+\lambda+\lambda^2+\dots\right) - i\left(\frac{\lambda n\pi}{2}\right)^2 + \dots$$

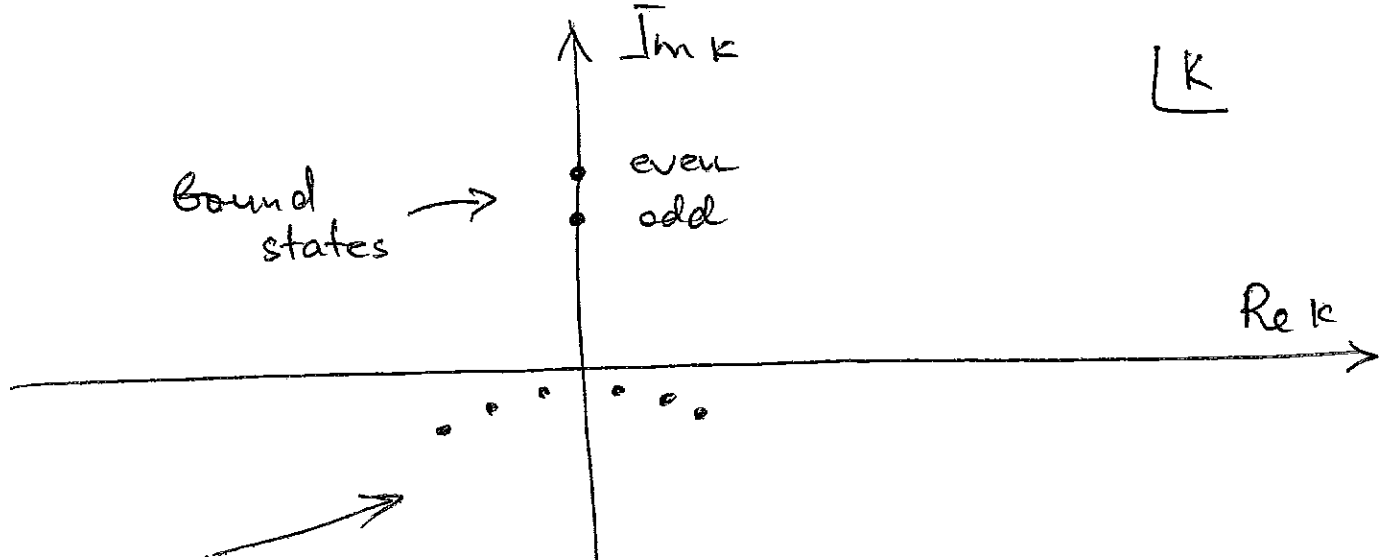
n odd. (These are poles of $F_+$.)

Similarly, solving Eq 2 one finds poles of $F_-$:

$$\xi_n^- = \frac{n\pi}{2}\left(1+\lambda+\lambda^2+\dots\right) - i\left(\frac{\lambda n\pi}{2}\right)^2 + \dots, \quad \underline{n \text{ even}}, \ n \neq 0$$

Note: a) approximation holds for $\lambda \ll 1$, $\lambda n \ll 1$

b) all these poles lie in the lower half-plane.




resonances (odd, even): symmetric pairs

$$F_+ = \frac{i\cos^2\xi}{\lambda\xi + \frac{1}{2}\sin 2\xi - i\cos^2\xi}$$

$$\xi = \frac{n\pi}{2}(1+\lambda+\dots) - i\left(\frac{n\pi\lambda}{2}\right)^2 + \dots \qquad n \text{ odd}$$

$$\cos\xi = -\sin\frac{n\pi}{2}\cdot\left(\xi - \frac{n\pi}{2}\right) + \dots$$

$$\sin\xi = \sin\frac{n\pi}{2} + \dots$$

$$\cos^2\xi = \left(\frac{n\pi\lambda}{2}\right)^2 + \dots$$

$$\lambda\xi + \frac{1}{2}\sin 2\xi - i\cos^2\xi = \lambda\frac{n\pi}{2} - \xi + \frac{n\pi}{2} - $$

$$- i\left(\frac{n\pi\lambda}{2}\right)^2 + \dots$$

$$F_+ = \frac{-i\left(n\pi\lambda/2\right)^2}{\xi - \frac{n\pi}{2} - \frac{n\pi\lambda}{2} + i\left(\frac{n\pi\lambda}{2}\right)^2} + \dots$$

$$F_+ = \frac{-i\,(n\pi\lambda/2)^2}{\xi - \xi_n^+} + \cdots \qquad \text{near } \xi = \xi_n^+$$

$$\xi_n^+ = \frac{n\pi}{2}(1+\lambda) - i\left(\frac{n\pi\lambda}{2}\right)^2$$

$$F_- = \frac{i \sin^2\xi}{\lambda\xi - \frac{1}{2}\sin 2\xi - i\sin^2\xi}$$

$$\sin^2\xi \approx \sin^2\frac{n\pi}{2} = 1 + \cdots$$

$$\lambda\xi - \frac{1}{2}\sin 2\xi = \lambda\frac{n\pi}{2} + \sin^2\frac{n\pi}{2}\left(\xi - \frac{n\pi}{2}\right) + \cdots$$

$$= \xi - \frac{n\pi}{2} + \frac{\lambda n\pi}{2} + \cdots \approx \lambda n\pi$$

$$F_- \simeq \frac{i}{\lambda n\pi - i} \approx -1$$

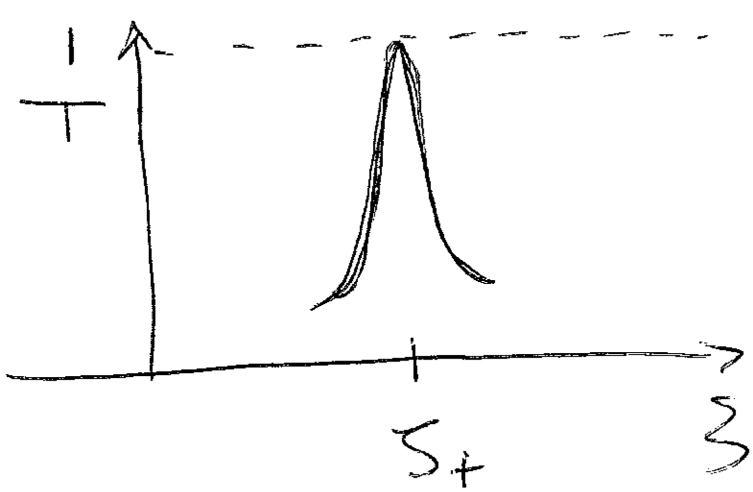
So, $S = 1 + F_+ + F_- \to$

$$\to -\frac{i\,(n\pi\lambda/2)^2}{\xi - \xi_n^+} = \frac{-i\,\bar{\eta}_+}{\xi - \zeta_+ + i\bar{\eta}_+} \quad \text{near } \Gamma$$

$$\zeta_+ \equiv \frac{n\pi}{2}(1+\lambda), \quad \bar{\eta}_+ \equiv (n\pi\lambda/2)^2 .$$

$$T = |S|^2 = \frac{\bar{\eta}_+^2}{(\xi - \zeta_+)^2 + \bar{\eta}_+^2}$$ on real $\xi$ axis, $\xi = ka$.

$\bar{\eta}_+ \ll 1$ Lorentzian shape:

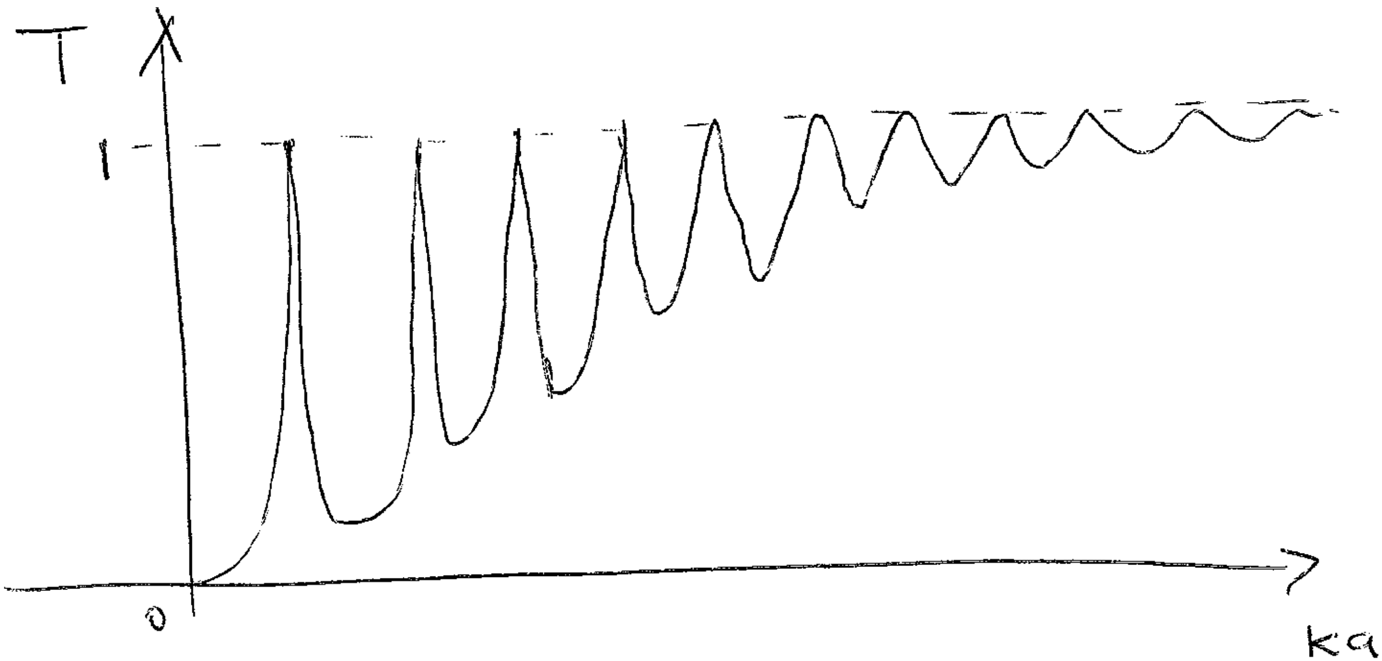


Also, $$R = 1 - \frac{\bar{\eta}_+^2}{(\xi-\zeta_+)^2 + \bar{\eta}_+^2} .$$

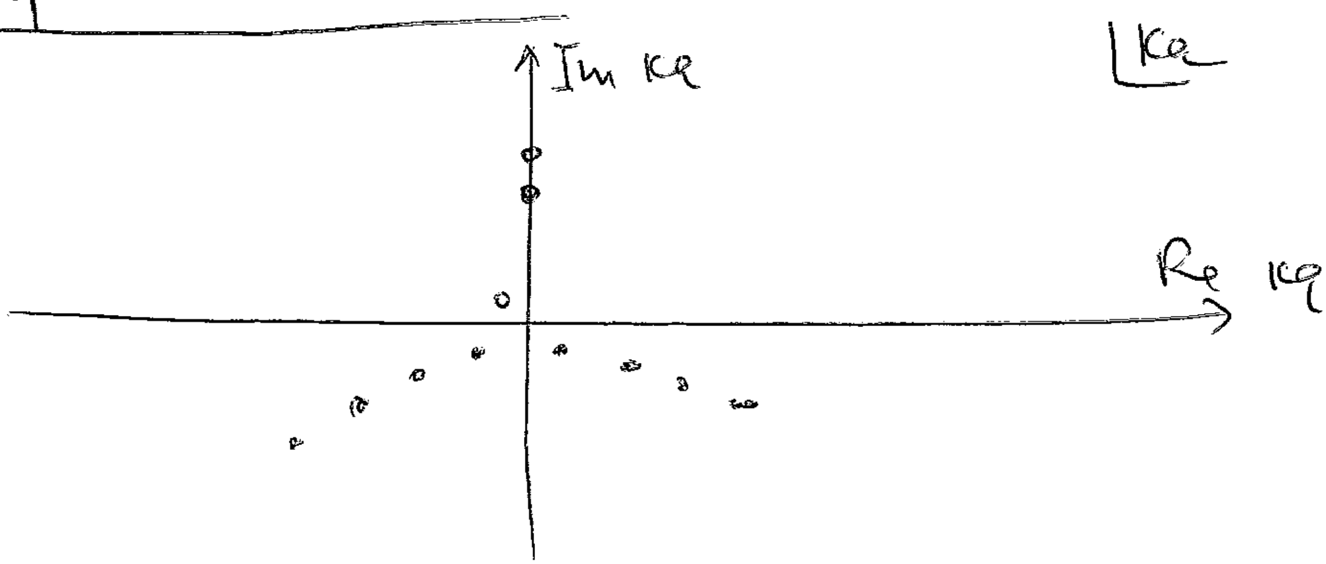
Thus: $\begin{cases} T \to 1 \\ R \to 0 \end{cases}$ at $\xi = S_+$

But near $\xi = S_+$ we have $\begin{matrix} R \to 1 \\ T \to 0 \end{matrix}$

<u>In total:</u>




<u>Compare with $S(k)$:</u>



In terms of energy:

$$E_n = \frac{\hbar^2 (\xi_n^+)^2}{2ma^2}$$

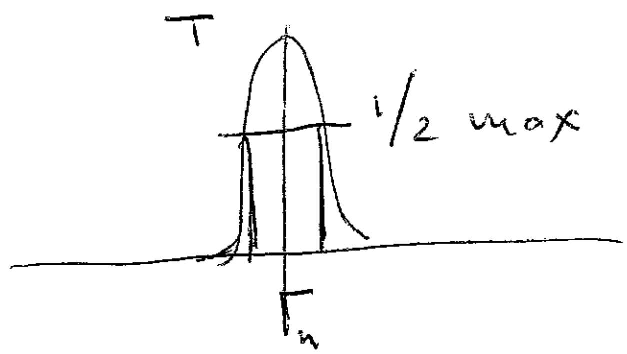
$$S(E) = -i \frac{\Gamma_n/2}{E - E_n + i\Gamma_n/2},$$

where $\Gamma_n \equiv \frac{2\hbar^2}{ma^2} \xi_n^+ \bar{\eta}_n^+ \simeq 2E_n n\pi \lambda^2 \ll E_n$

$$T = |S|^2 = \frac{\Gamma_n^2/2}{(E-E_n)^2 + \Gamma_n^2/4} \quad (\text{near } E = E_n)$$

Breit-Wigner

FWHM: $T = 1/2$ when width at half max $= \Gamma_n$



We have seen that the resonances are present as poles in the complex $k$-plane of $S(k)$:

$$K = \pm\alpha - i\beta, \quad \alpha, \beta > 0,$$

omitting all constants.

$$\sqrt{E} = |E|^{1/2} e^{i\varphi/2 + in\pi} = K = \pm\alpha - i\beta$$

$$\pm |E|^{1/2}\left(\cos\varphi/2 + i\sin\varphi/2\right) = \pm\alpha - i\beta$$

$n = 0, 1$

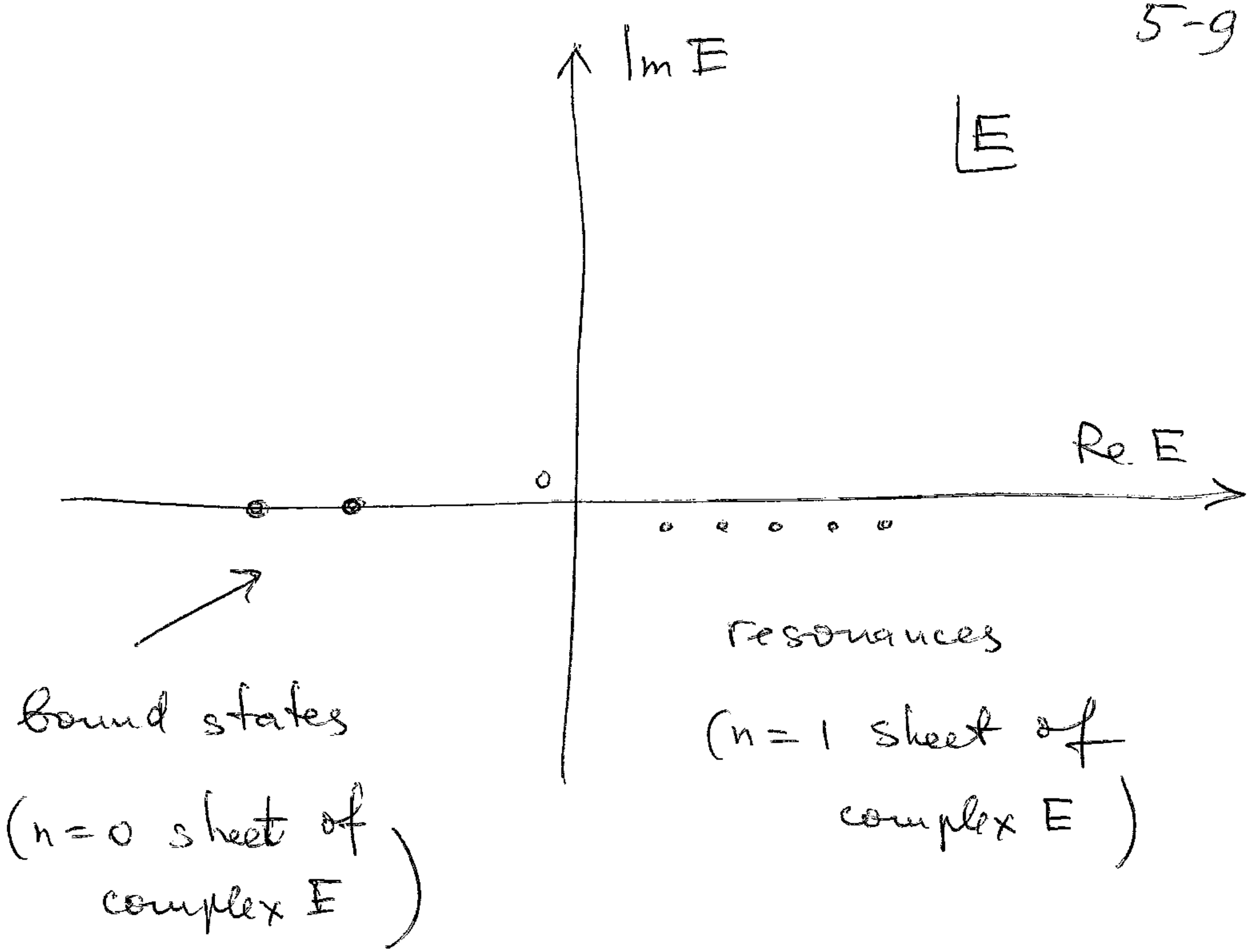
$\varphi \in [0, 2\pi]$ on the phys. sheet $\left(E = |E| e^{i\varphi}\right)$

$\Rightarrow \varphi/2 \in [0, \pi] \Rightarrow \sin\varphi/2 > 0$ on the phys. sheet $\Rightarrow$ <u>must choose $n = 1$</u> since $\beta > 0$.

$$\Rightarrow -|E|^{1/2}\sin\varphi/2 = -\beta \qquad \text{OK};$$

$$-|E|^{1/2}\cos\varphi/2 = \pm\alpha \qquad \text{OK}.$$

Moral: resonances are on the $n=1$ sheet of complex $E$.


Im E
E
Re E
0
bound states
(n = 0 sheet of
complex E)
resonances
(n = 1 sheet of
complex E)

Remarks:

1. In addition to poles on the positive Im axis of $k$ (bound states) and pairs of poles in the lower half-plane, $S(k)$ may have other singularities:

- „false" poles on $\mathrm{Im}\, k > 0$
- virtual (anti-bound) states on $\mathrm{Im}\, k < 0$

False poles do not correspond to bound states. They do not appear for $U(x)$ vanishing at $|x| \to \infty$ faster than $e^{-\alpha|x|}$.
In particular, they do not appear for potential well or $\delta$-function pot., where $U(x) = 0$ at $x \to \pm\infty$.

To get rid of false poles, one can cut the pot. at some $|x| = L$, and then ~~take~~ $L \to \infty$.

<u>Virtual states</u>: recall $U(x) = -q\,\delta(x)$,

$$S(k) = \frac{ik}{ik + \varkappa}, \quad \text{where } q, \varkappa > 0.$$

Now consider $q, \varkappa < 0$ (repulsive pot)

The pole of $S$ is at $k = k_* = i\varkappa$, i.e. in the lower half-plane. Virtual states may affect $T(E)$ and $R(E)$.

<u>Moral</u>: $S(k)$ can have finite or infinite number of poles of 4 types:

- on the positive $\operatorname{Im} k$ (Bound states): $N_b$
- on the positive $\operatorname{Im} k$ (false poles): $N_f$
- on the negative $\operatorname{Im} k$ (virtual states) $N_v$
- pairs in the lower half-plane (resonances) $N_r$

2. There are formulas relating $N_b, N_f, N_v, N_r$ and $\delta_\pm(k)$.
These are known as versions of Levinson's theorem (N. Levinson, 1949).

3. Naively, one can think of resonances as states with $E = E_n - i\Gamma_n/2 \Rightarrow$
$\psi \sim e^{-iEt/\hbar} \sim e^{-iE_n t/\hbar} e^{-\Gamma t/\hbar}$ i.e.
as quasi-stationary states with life-time $\tau \sim \hbar/\Gamma$. But the Hamiltonian is Hermitian $\Rightarrow$ can't have complex $E$!
Need to treat $\psi$ as wave packets

$$\psi(x,t) = \int dk\, A(k)\, e^{-\frac{iE_k}{\hbar}t + ikx}$$

and consider time-dep. Schrödinger eq.