Lecture 4

Last time, for the potential

$U(x) = -\gamma\left[\delta(x-a) + \delta(x+a)\right]$ we found

$S(E) = 1 + F_+ + F_-$ and $A(E) = F_+ - F_-$,

where

$$F_\pm = \frac{iY_\pm}{X_\pm - iY_\pm} = \frac{1}{2}\left(e^{2i\delta_\pm} - 1\right)$$

$$X_\pm = \lambda\xi \pm \frac{1}{2}\sin 2\xi \qquad Y_\pm = \begin{cases} \cos^2\xi \\ \sin^2\xi \end{cases}$$

Note that: $S(E) = \frac{1}{2}\left(e^{i2\delta_+} + e^{i2\delta_-}\right)$.

$$A(E) = \frac{1}{2}\left(e^{i2\delta_+} - e^{i2\delta_-}\right).$$

We expect the bound states of $U(x)$ to correspond to poles of $S(k)$ on the positive Im axis of complex k (for S(E) - to poles on the negative real axis of complex E).

We also want to check for other singularities of S(k) (or S(E)) and understand their physical meaning.

I. Bound states of U(x)

Recall our integral eq. for $\psi(x)$:

$$\psi(x) = -\frac{m}{\varkappa \hbar^2} \int_{-\infty}^{\infty} e^{-\varkappa |x-x'|} U(x') \psi(x') \, dx',$$

$\varkappa = \sqrt{2m|E|}/\hbar > 0$. Subst. U(x):

$$\psi(x) = \frac{\gamma m}{\varkappa \hbar^2} \left[e^{-\varkappa |x+a|} \psi(-a) + e^{-\varkappa |x-a|} \psi(a) \right],$$

i.e.
$$\psi(a) = \frac{\gamma m}{\varkappa \hbar^2}\left[e^{-\varkappa 2a}\psi(-a) + \psi(a)\right]$$

$$\psi(-a) = \frac{\gamma m}{\varkappa \hbar^2}\left[\psi(-a) + e^{-\varkappa 2a}\psi(a)\right]$$

or

$$\begin{bmatrix} 1 - \frac{\bar\gamma}{2\varkappa} & -\frac{\bar\gamma}{2\varkappa} e^{-\varkappa 2a} \\ -\frac{\bar\gamma}{2\varkappa} e^{-2a\varkappa} & 1 - \frac{\bar\gamma}{2\varkappa} \end{bmatrix} \begin{pmatrix} \psi(a) \\ \psi(-a) \end{pmatrix} = 0$$

This has non-triv. solution iff:

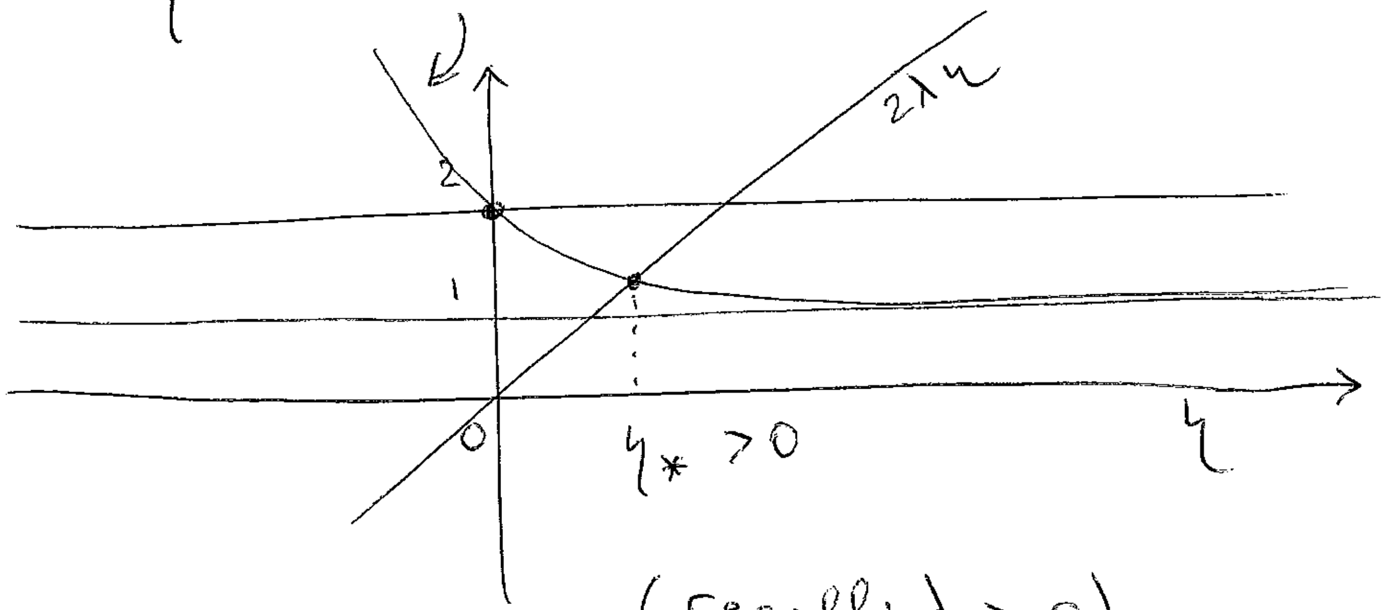
$$\left(1 - \frac{\bar\gamma}{2\varkappa}\right)^2 - \left(\frac{\bar\gamma}{2\varkappa}\right)^2 e^{-4\varkappa a} = 0 \,; \quad \bar\gamma = \frac{\gamma 2m}{\hbar^2}$$

$$\lambda = \frac{1}{\bar\gamma a}$$

or $\quad 2\lambda \varkappa a = 1 \pm e^{-2\varkappa a}$ $\quad$ or

$$2\lambda\eta = 1 \pm e^{-2\eta}, \qquad \eta \equiv \varkappa a .$$

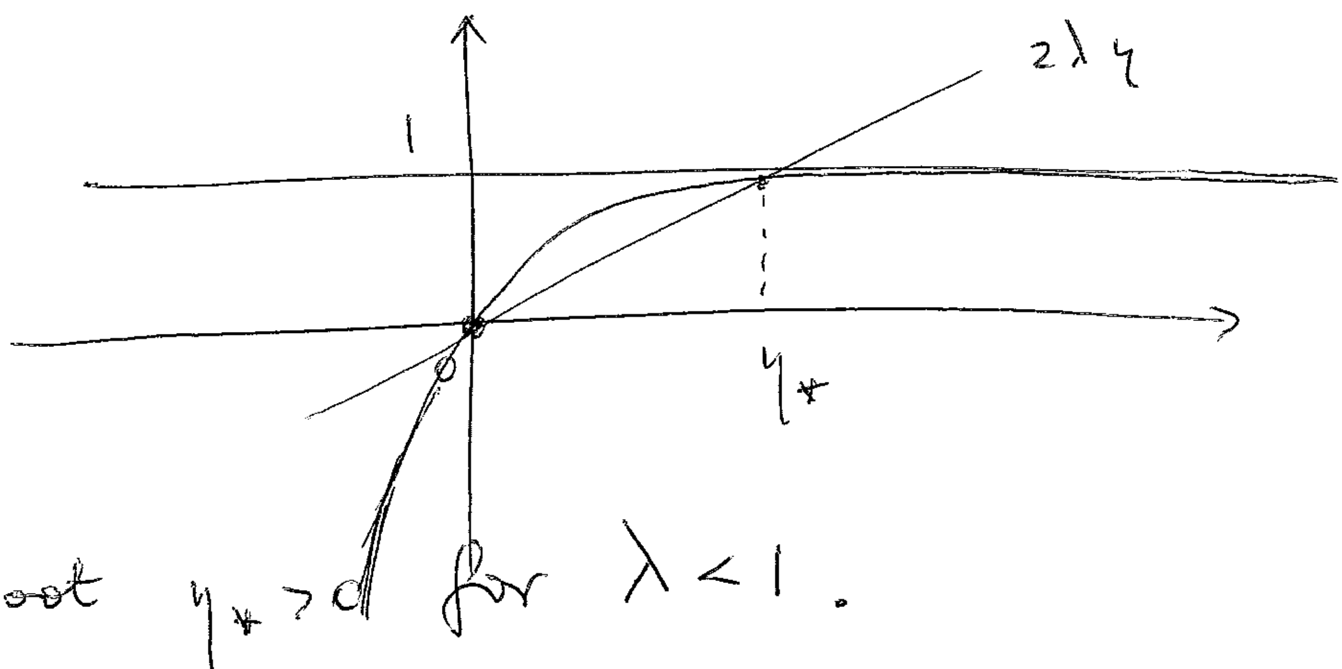
1) $2\lambda\eta = 1 + e^{-2\eta}$



(recall: $\lambda > 0$)

One root $\eta_* > 0$ for $\forall \lambda > 0$.

2) $2\lambda\eta = 1 - e^{-2\eta}$



One root $\eta_* > 0$ for $\lambda < 1$.

These are even / odd parity bound states
($\psi(a) = \pm\psi(-a)$)

This can be analysed further using approximations such as $\lambda \to 0$:

$\underline{\lambda \to 0}$: η_* is large $\to$ exp are small

and $\eta_* \approx \frac{1}{2\lambda}$ for both even/odd states. Next approximation:

$$\eta_* \approx \frac{1}{2\lambda}\left(1 \pm e^{-1/\lambda}\right)$$

or $$E^{\pm}_{even/odd} \simeq -\frac{m\gamma^2}{2\hbar^2}\left(1 \pm e^{-2am\gamma/\hbar^2}\right).$$

$\underline{\lambda \to \infty}$: only one (even) level with

$$E^{+}_{even} \simeq -\frac{2m\gamma^2}{\hbar^2}.$$

Now look at singularities of $S(k)$:
zeros of denominators of $F_\pm(\xi)$. Eqs:

$$\lambda\xi + \frac{1}{2}\sin 2\xi - i\cos^2\xi = 0 \qquad (1)$$

$$\lambda\xi - \frac{1}{2}\sin 2\xi - i\sin^2\xi = 0 \qquad (2)$$

Treat ξ as complex var. with $\mathrm{Re}\,\xi = ka$,

$\xi = \zeta + i\eta$.

<u>Use</u>: $\sin(x \pm iy) = \sin x\cosh y \pm i\cos x\sinh y$

$\cos(x \pm iy) = \cos x\cosh y \mp i\sin x\sinh y$

$\cosh x - \sinh x = e^{-x}$

<u>Separate Real and Im parts.</u>

Eq (1) gives:

$$\begin{cases} 2\lambda\zeta + \sin 2\zeta\, e^{-2\eta} = 0, \\ 2\lambda\eta - \cos 2\zeta\, e^{-2\eta} = 1. \end{cases}$$

and Eq (2) gives

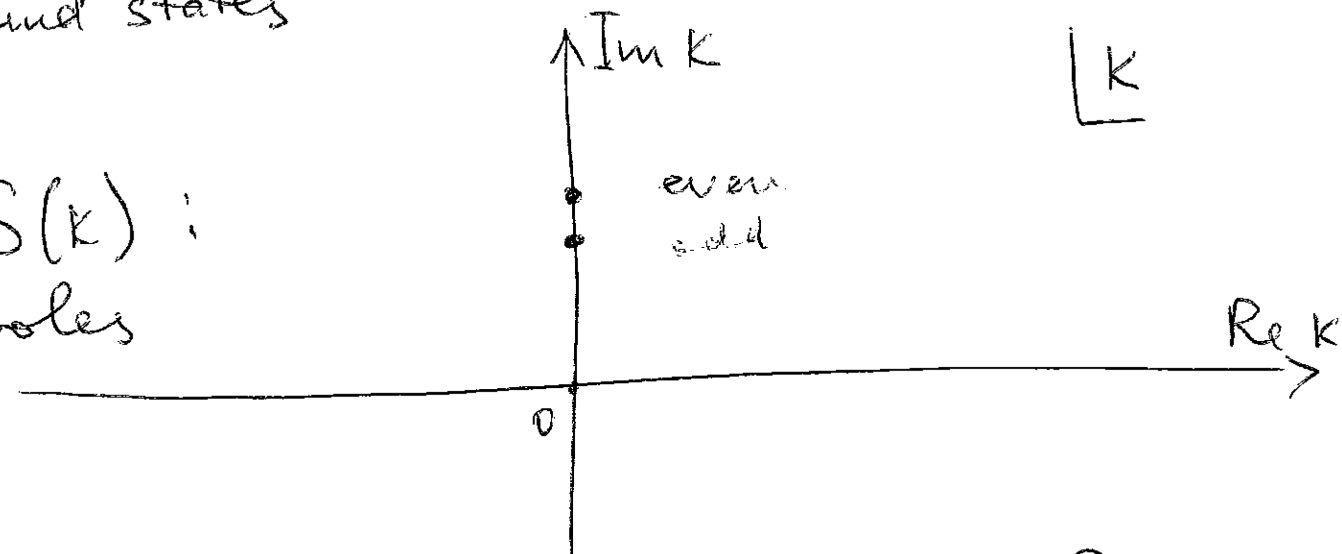
$$\begin{cases} 2\lambda\xi - \sin 2\xi \, e^{-2\eta} = 0, \\ 2\lambda\eta + \cos 2\xi \, e^{-2\eta} = 1. \end{cases}$$

For $\xi = 0$ (on the imaginary k-axis) we have:

$$\boxed{2\lambda\eta = 1 \pm e^{-2\eta}} \qquad \eta = \operatorname{Im} ka$$

i.e. precisely the eqs that determine the bound states

$S(k)$: poles



other singularities – ?

Eqs (1), (2) can be solved numerically.
But let us look at $\lambda \to 0$ case.

Eq (1):

$$\begin{cases} \sin 2\xi = -2\lambda \xi e^{2\eta} \to 0 \\ \cos 2\xi = 2\lambda \eta e^{2\eta} - e^{2\eta} \to -e^{2\eta} \end{cases}$$

$$2\xi \approx n\pi, \quad n = 0, \pm 1, \dots$$

$$\cos 2\xi = (-1)^n \approx -e^{2\eta} \Rightarrow n \text{ odd}, \quad \eta \approx 0.$$

So, to leading order in λ:

$$\begin{cases} \xi_n^+ = \frac{n\pi}{2}, \\ \eta_n^+ = 0, \end{cases} \quad n = \pm 1, \pm 3, \dots$$

Next order: $X = X_n + \varepsilon, \quad X \equiv 2\xi$

$$\sin X_n + \cos X_n \cdot \varepsilon + \ldots = -\lambda X_n \cancel{e^{2y}}$$

$$-\varepsilon = -\lambda n\pi$$

$$\begin{cases} \xi_n^+ = \xi_n^{+(0)} + \lambda \dfrac{n\pi}{2} = \dfrac{n\pi}{2}(1+\lambda) + \cdots \\ \eta_n^+ = 0 - \left(\dfrac{\lambda n\pi}{2}\right)^2 + \cdots \end{cases}$$

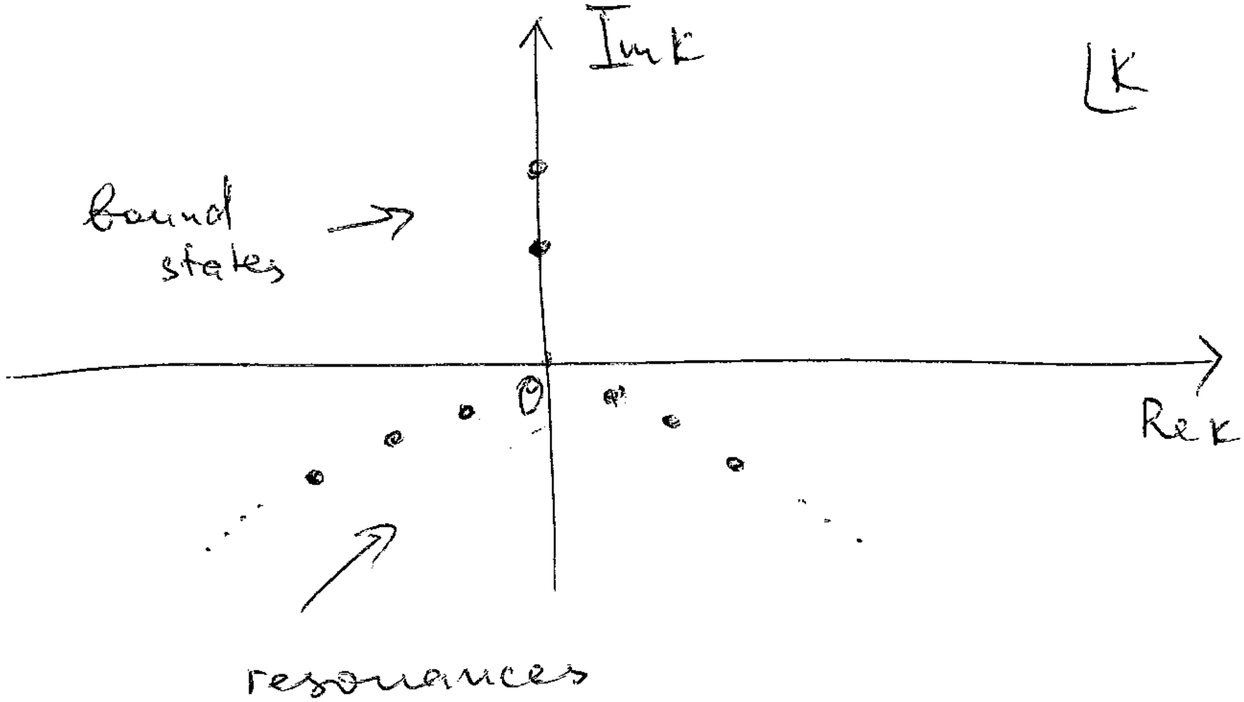
Similarly, Eq (2) lead to

$$\begin{cases} \zeta_n^- = \dfrac{n\pi}{2}(1+\lambda) + \cdots, \quad n = 0, \pm 2, \pm 4 \\ \eta_n^- = -\left(\dfrac{\lambda n\pi}{2}\right)^2 + \cdots \end{cases}$$

So, $\xi = \zeta + i\eta = \dfrac{n\pi}{2}(1+\lambda+\ldots) - i\left(\dfrac{\lambda n\pi}{2}\right)^2 + \ldots$

for $\lambda \ll 1$, with n = odd or even.

<u>Note</u>: $\operatorname{Im} \xi < 0$.



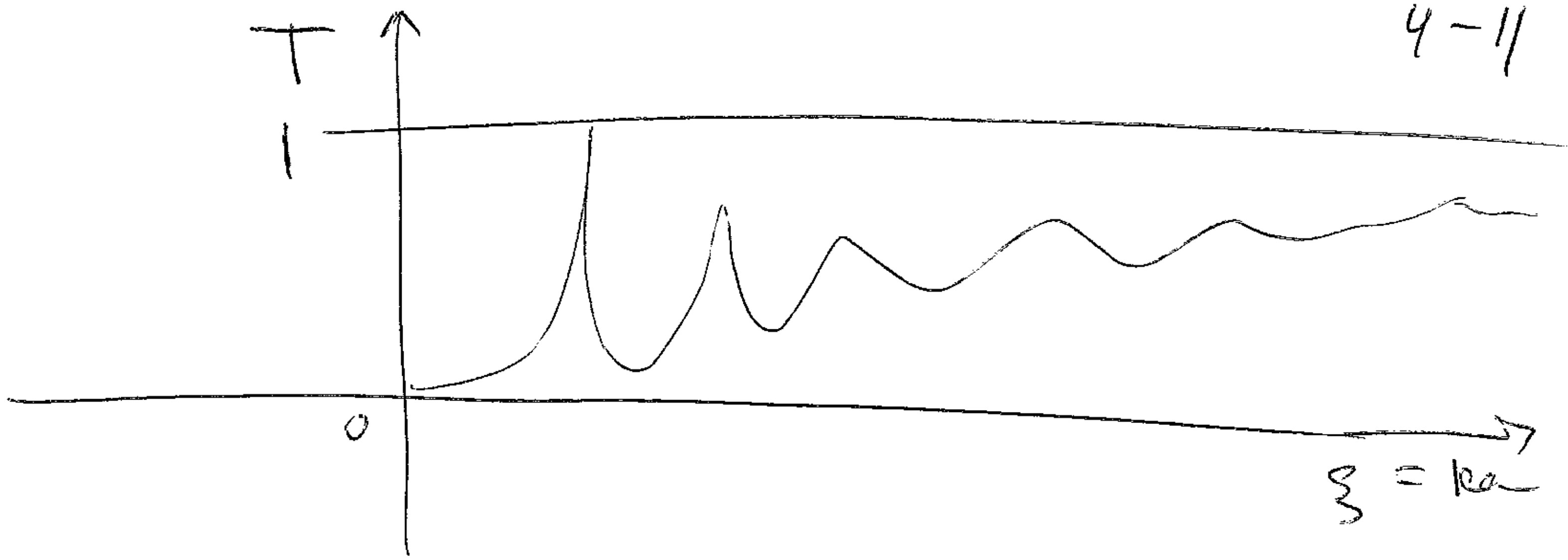
Recall: $S = 1 + F_+ + F_-$

e.g. for ξ_n^+:

$$F_+ \approx \frac{-i\eta_n^+}{-(\xi-\xi_n^+)+i\eta_n^+}, \quad F_- \approx -1$$

$$\Rightarrow S(k) = \frac{i\eta_n^+}{\xi-\xi_n^+-i\eta_n^+}$$

$$T = |S|^2 = \frac{\eta_n^2}{(\xi-\xi_n^+)^2+\eta_n^2}, \quad |\eta_n| \ll 1.$$



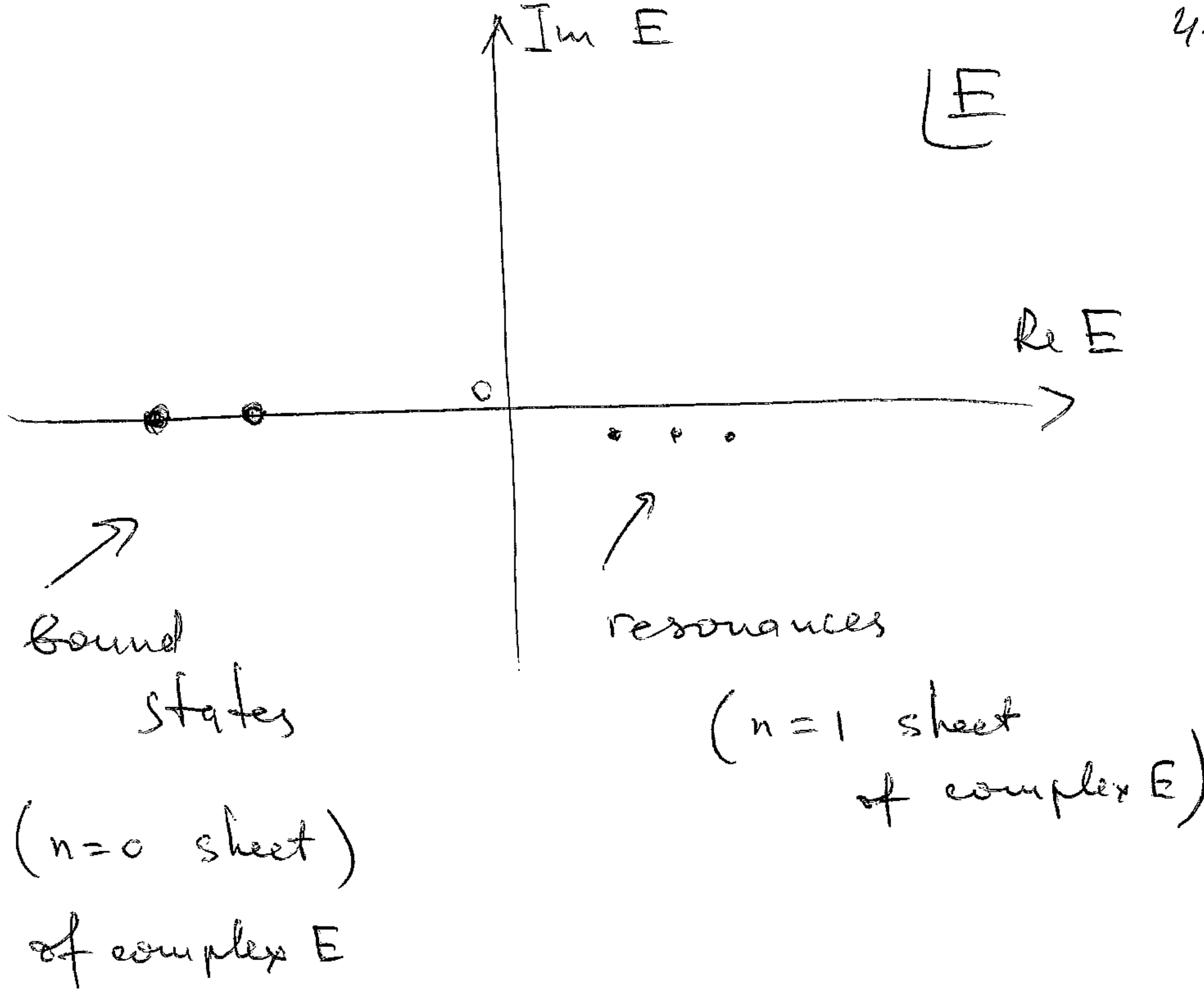
$$E_n = \frac{\hbar^2 (\xi_n^+)^2}{2ma^2} \simeq E_n^{(0)} (1 + 2\lambda)$$

$$S(E) = -i \frac{\Gamma_n/2}{E - E_n + i\Gamma_n/2}$$

$$\Gamma_n = \frac{2\hbar^2}{ma^2} \xi_n^+ |\eta_n| > 0 \simeq 2E_n n\pi\lambda^2 \ll E_n .$$

$$T = |S|^2 = \frac{\Gamma_n^2/2}{(E - E_n)^2 + \Gamma_n^2/4}$$ (Breit-Wigner)

FWHM: $T = 1/2$ when width $= \Gamma_n$.


Im E
E
Re E
0
bound states
(n=0 sheet)
of complex E
resonances
(n=1 sheet
of complex E)