Lecture 3

$$\psi_+(a) = \cos ka + \gamma\left[G^+(a,-a) + G^+(a,a)\right]\psi_+(a)$$

$$G^+(x,x') = \frac{im}{k\hbar^2}\exp\left(ik|x-x'|\right)$$

$$G^+(a,a) = \frac{im}{k\hbar^2}$$

$$G^+(a,-a) = \frac{im}{k\hbar^2}e^{i2ka}$$

$$\psi_+(a) = \cos ka + \frac{i\gamma m}{k\hbar^2}\left(1+e^{i2ka}\right)\psi_+(a)$$

$$\psi_+(a) = \frac{\cos ka}{1 + \frac{\gamma m}{ik\hbar^2}\left(1+e^{i2ka}\right)} =$$

$$= \frac{\cos ka}{1 + \frac{\bar{\gamma}}{2ik}\left(1+e^{i2ka}\right)} \equiv \frac{\cos ka}{1+\bar{\gamma}K_+},$$

where $\bar{\gamma} \equiv \gamma\, 2m/\hbar^2$.

$$\psi_-(a) = i \sin ka + \gamma \left(G^+(a,a) - G^+(a,-a) \right) \psi_-(a)$$

$$\psi_-(a) = i \sin ka + \frac{i m \gamma}{k \hbar^2} \left(1 - e^{i2ka} \right) \psi_-(a)$$

$$\psi_-(a) = \frac{i \sin ka}{1 + \frac{\bar{\gamma}}{2ik}\left(1 - e^{i2ka}\right)} = \frac{i \sin ka}{1 + \bar{\gamma} K_-},$$

$$K_\pm = \frac{1}{2ik}\left(1 \pm e^{i2ka}\right).$$

$$\psi(x) = \psi_+(x) + \psi_-(x) = e^{ikx} +$$

$$+ \frac{i \gamma m}{k \hbar^2} \left(e^{ik|x-a|} + e^{ik|x+a|} \right) \psi_+(a) +$$

$$+ \frac{i \gamma m}{k \hbar^2} \left(e^{ik|x-a|} - e^{ik|x+a|} \right) \psi_-(a)$$

$$\psi(x) = e^{ikx} - \frac{\bar{\gamma}}{2ik}\left\{\left(e^{ik|x-a|} + e^{ik|x+a|}\right)\psi_+(a) + \right.$$

$$\left. + \left(e^{ik|x-a|} - e^{ik|x+a|}\right)\psi_-(a)\right\}$$

Find first A in $e^{ikx} + Ae^{-ikx}$ for $x < -a$, and $R = |A|^2$:

Note: $x + a < 0$ and $x - a < -2a < 0$.

The coefficient in front of e^{-ikx} is:

$$A = -\frac{\bar{\gamma}}{ik}\left[\cos ka\, \psi_+(a) + i \sin ka\, \psi_-(a)\right] =$$

$$= \frac{i\bar{\gamma}}{k}\left[\frac{\cos^2 ka}{1+\bar{\gamma}K_+} - \frac{\sin^2 ka}{1+\bar{\gamma}K_-}\right].$$

Similarly,

$$S(k) = 1 + \frac{i\bar{\gamma}}{k}\left[\frac{\cos^2 ka}{1+\bar{\gamma}K_+} + \frac{\sin^2 ka}{1+\bar{\gamma}K_-}\right].$$

ex: Show that $R + T = 1$.

Introduce dimensionless var:

$$\xi = \kappa a$$
$$\lambda = 1/\bar{\gamma} a$$

Then: $$S(E) = 1 + F_+(\xi) + F_-(\xi)$$
$$A(E) = F_+(\xi) - F_-(\xi),$$

where $$F_+ = \frac{i\cos^2\xi}{\lambda\xi + \frac{1}{2}\sin 2\xi - i\cos^2\xi}$$

$$F_- = \frac{i\sin^2\xi}{\lambda\xi - \frac{1}{2}\sin 2\xi - i\sin^2\xi}$$

Note: $\lambda\xi = \frac{\kappa a}{\bar{\gamma} a} \to 0$ for $\bar{\gamma} \to \infty$

In this limit, $F_+ \to -e^{-i\xi}\cos\xi$

$F_- \to -i\,e^{-i\xi}\sin\xi \Rightarrow S(E) \to 0$

and $T \to 0$ $(R \to 1)$ unless

$\cos\xi = 0$ in F_+ and $\sin\xi = 0$ in F_-.

But these are conditions for bound states of even / odd parity in an infinite pot. well.

Consider further $F_\pm$:

$$F_\pm = \frac{i\,Y_\pm}{X_\pm - i\,Y_\pm} = \frac{1}{2}\left(e^{2i\delta_\pm} - 1\right)$$

$$X_\pm = \lambda\xi \pm \tfrac{1}{2}\sin 2\xi \qquad Y_\pm = \begin{cases} \cos^2\xi \\ \sin^2\xi \end{cases}$$

Indeed, $z_\pm = \dfrac{Y_\pm}{X_\pm - iY_\pm} =$

$$= \frac{Y_\pm (X_\pm + iY_\pm)}{X_\pm^2 + Y_\pm^2} = \frac{X_\pm Y_\pm + i Y_\pm^2}{X_\pm^2 + Y_\pm^2}$$

If $z_\pm = \rho_\pm e^{i\delta_\pm} \Rightarrow \rho_\pm^2 = |z_\pm|^2 = \dfrac{Y_\pm^2}{X_\pm^2 + Y_\pm^2} = \operatorname{Im} z_\pm$

$\Rightarrow \rho^2 = \rho \sin\delta \Rightarrow \rho = \sin\delta$

$\sin\delta = \dfrac{Y}{\sqrt{X^2+Y^2}}$ and $z = e^{i\delta}\sin\delta =$

$$= \frac{e^{i2\delta} - 1}{2i}.$$

Thus $F_\pm = i z_\pm = \frac{1}{2}\left(e^{i2\delta_\pm} - 1\right).$

$$F_\pm = \frac{1}{2}\left(e^{i2\delta_\pm} - 1\right) = i e^{i\delta_\pm} \sin\delta_\pm .$$

$$S(E) = 1 + F_+ + F_- = \frac{1}{2}\left(e^{i2\delta_+} + e^{i2\delta_-}\right).$$

$$A(E) = F_+ - F_- = \frac{1}{2}\left(e^{i2\delta_+} - e^{i2\delta_-}\right).$$

Now look at the poles of $S(E)$:

e.g. $\lambda\xi + \frac{1}{2}\sin 2\xi - i\cos^2\xi = 0$

<u>Simplify by considering $\lambda \ll 1$ $(\bar{\gamma} a \gg 1)$</u>

Then: $\lambda\xi + \frac{1}{2}\sin 2\xi \approx 0$

$$\frac{\sin 2\xi}{2\xi} \approx \lambda \ll 1 \Rightarrow \xi_n^+ = \frac{n\pi}{2}, \quad n = 0, \pm 1 \ldots$$

<u>Improve</u>: $\xi = \xi_n + \varepsilon$ (n odd)

$$\frac{2\cos 2\xi_n^+ (\xi - \xi_n^+)}{2(\xi_n^+ + \varepsilon)} = \lambda \Rightarrow \xi_n^+ = \frac{n\pi}{2}\left(1 + \lambda + O(\lambda^2)\right)$$

$$\cos^2 \xi = \frac{1}{2}\left(1 + \cos 2\xi\right)$$

$$\cos\left[n\pi(1+\lambda)\right] = -1 + \frac{(n\pi\lambda)^2}{2}$$

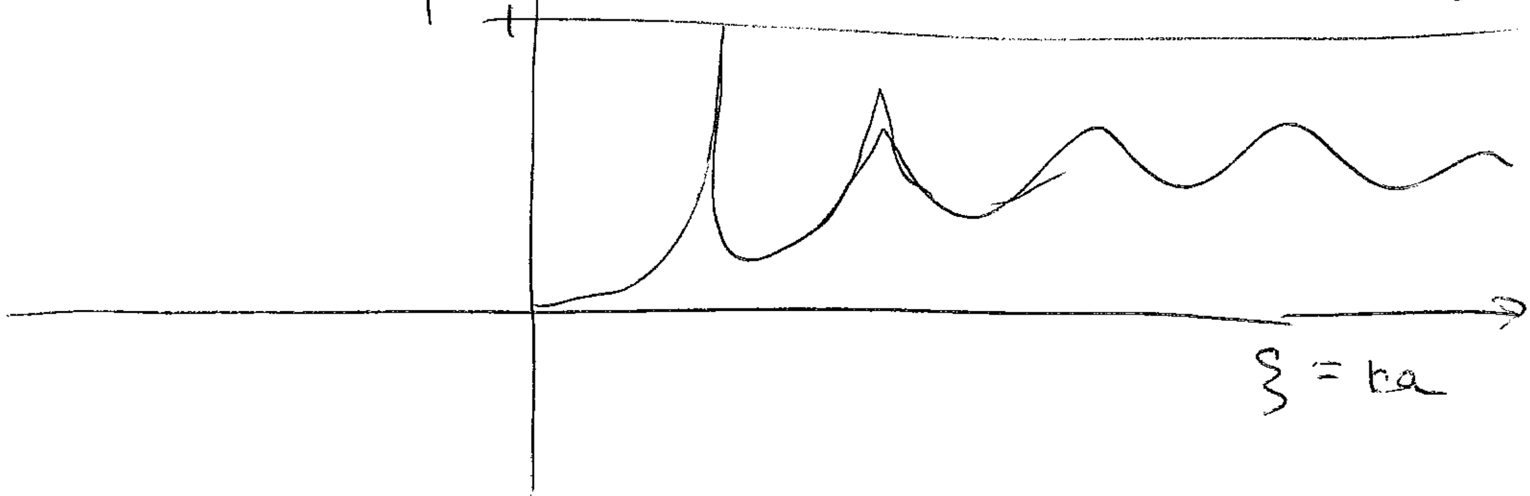
$$\cos^2 \xi = \left(\frac{n\pi\lambda}{2}\right)^2 + \cdots = O(\lambda^2) \equiv \eta_n$$

$$F_+ \approx \frac{i\,\eta_n}{-(\xi - \xi_n^+) - i\,\eta_n}\,, \qquad F_- \approx -1$$

$$S(E) = 1 + F_+ + F_- \approx F_+$$

$$S(E) \approx \frac{-i\,\eta_n}{\xi - \xi_n^+ + i\,\eta_n}$$

$$T = |S|^2 = \frac{\eta_n^2}{(\xi - \xi_n^+)^2 + \eta_n^2}\,, \qquad \underline{\eta_n \ll 1}$$



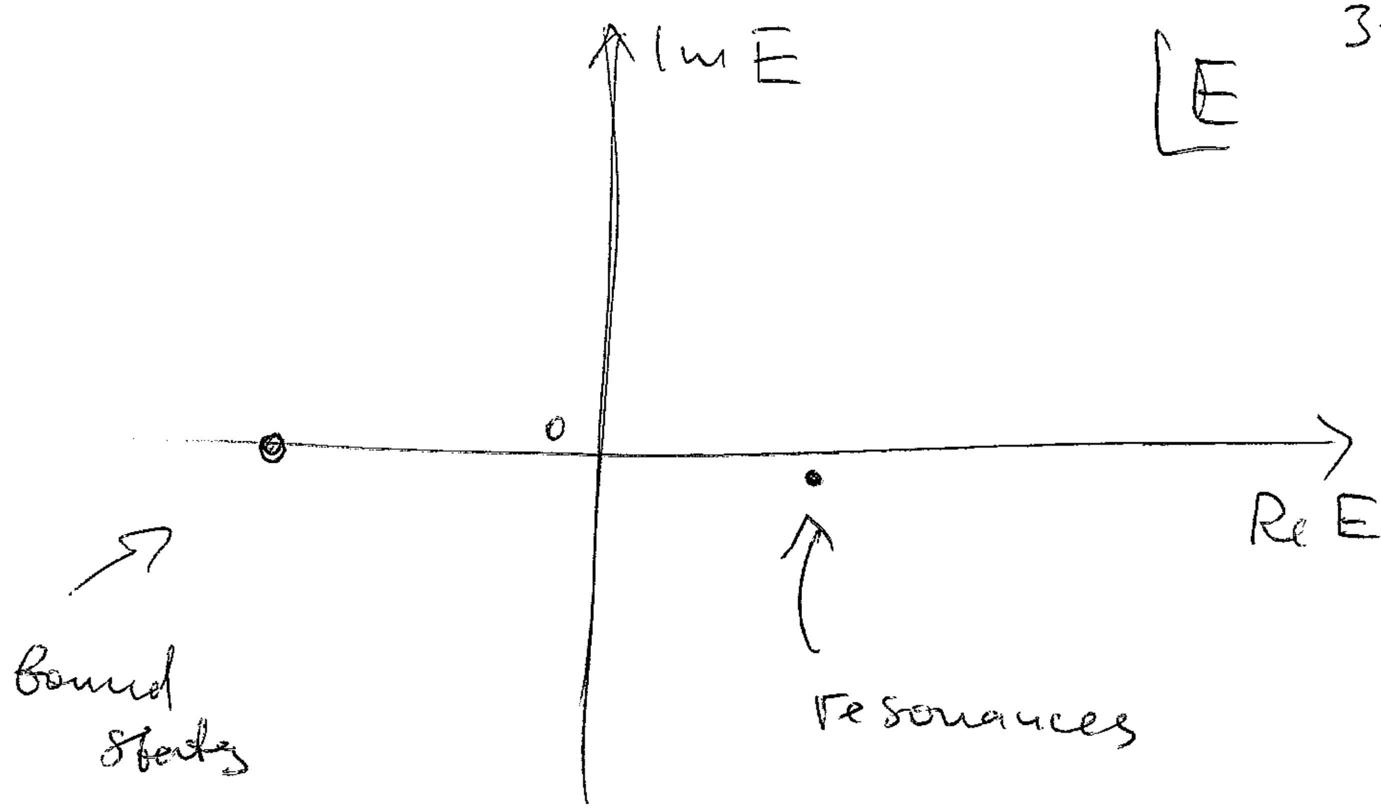
$$E_n = \frac{\hbar^2 (\xi_n^+)^2}{2ma^2} \cong E_n^{(0)} (1 + 2\lambda).$$

$$S(E) = -i \, \frac{\Gamma_n/2}{E - E_n + i\Gamma_n/2},$$

$$\Gamma_n = \frac{2\hbar^2}{ma^2} \xi_n^+ \eta_n \simeq 2 E_n n\pi \lambda^2 \ll E_n.$$

$$T = |S|^2 = \frac{\Gamma_n^2/2}{(E - E_n)^2 + \Gamma_n^2/4}$$

FWHM : $T = 1/2$: width $= \Gamma_n$. Breit-Wigner


E
Im E
0
Re E
Bound States
resonances