Lecture 10

The Dirac equation

Non-relativ. electron (with spin) in external field with $A^{\mu} = (\Phi, \vec{A})$ obeys Pauli eq:

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{\left(\hat{\vec{p}} - \frac{e}{c}\vec{A}\right)^2}{2m} - \mu \vec{\sigma}\cdot\vec{B} + e\Phi \right] \varphi ,$$

where $\vec{B} = \operatorname{curl} \vec{A}$, $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$,

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices, $\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$ is a two-component spinor:

$$\hat{\vec{S}} = \frac{\hbar}{2}\vec{\sigma} \qquad \begin{cases} \hat{S}_z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \hat{S}_z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\sigma_i \sigma_j = \delta_{ij} + i \varepsilon_{ijk} \sigma_k$$

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}$$

Need Lorentz-covar. eq. with $\frac{\partial \psi}{\partial t}$.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1 \frac{\partial \psi}{\partial x^1} + \alpha_2 \frac{\partial \psi}{\partial x^2} + \alpha_3 \frac{\partial \psi}{\partial x^3}\right) +$$

$$+ \beta m c^2 \psi \equiv \hat{H}_D \psi. \qquad (*)$$

α_i, β cannot be numbers (otherwise the eq. is not even rot. invar.) $\Rightarrow$ <u>matrices</u>

$\alpha_i \alpha_j \neq \alpha_j \alpha_i$

<u>Conditions:</u>

1. need to recover $E^2 = p^2c^2 + m^2c^4$
2. prob. interpr. of ψ and continuity eq: $\partial \rho / \partial t + \operatorname{div} \vec{j} = 0$
3. eq should be Lorentz-covariant
4. $\hat{H}$ must be Hermitian

$$\psi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_N \end{pmatrix}$$

Condition 1 satisfied if each ψ_α satisfies the KG eq

$$-\hbar^2 \frac{\partial^2 \psi_\alpha}{\partial t^2} = \left(-\hbar^2 c^2 \vec{\nabla}^2 + m^2 c^4\right)\psi_\alpha, \quad \alpha = 1, \dots N$$

Square the eq:

$$i\hbar \partial_t \psi = \hat{H}_D \psi$$

$$-\hbar^2 \partial^2_{tt} \psi = i\hbar \hat{H}_D \frac{\partial \psi}{\partial t} = \hat{H}_D^2 \psi \,;$$

$$-\hbar^2 \frac{\partial^2 \psi}{\partial t^2} = -\hbar^2 c^2 \sum_{i,j=1}^{3} \frac{\alpha_j \alpha_i + \alpha_i \alpha_j}{2} \frac{\partial^2 \psi}{\partial x^i \partial x^j} +$$

$$+ \frac{\hbar m c^3}{i} \sum_{i=1}^{3} (\alpha_i \beta + \beta \alpha_i) \frac{\partial \psi}{\partial x_i} + \beta^2 m^2 c^4 \psi$$

$\Rightarrow$ KG for ψ_α if

$$\boxed{\alpha_i^2 = \mathbb{1}, \quad \beta^2 = \mathbb{1}, \quad \begin{array}{l} \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik} \\ \alpha_i \beta + \beta \alpha_i = 0 \end{array}}$$

$\Rightarrow$ eigenvalues of α_i, β are ± 1.

~~Proof~~ Proof $\rightarrow$

Indeed,

$$\alpha_i \varphi = \lambda \varphi$$

$$\underbrace{\alpha_i^2 \varphi}_{= \varphi} = \lambda \alpha_i \varphi = \lambda^2 \varphi$$

$\Rightarrow \lambda^2 = 1$ But α_i Hermitian $\Rightarrow$ eigenvalues real $\Rightarrow \lambda = \pm 1$. Same for β.

Also, $\operatorname{tr} \alpha_i = 0$, $\operatorname{tr} \beta = 0$.

Indeed: $\alpha_i = -\beta \alpha_i \beta$

$$\operatorname{tr} \alpha_i = -\operatorname{tr}(\beta \alpha_i \beta) = -\operatorname{tr} \alpha_i .$$

Since $\operatorname{tr} \alpha_i = \sum_{p=1}^{N} \lambda_p = +1 + 1 \ldots -1 -1$

$\Rightarrow$ N even, $N > 2$ ($N = 2$ only 3 Pauli matrices, need 4).

$N = 4$: we can take e.g.

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Why $N=4$?

$\alpha_i^2 = \mathbb{1}$ $\quad \beta^2 = \mathbb{1}$ $\quad \alpha_i \alpha_k + \alpha_k \alpha_i = 2\delta_{ik}$

α_i, β : Hermitian $\quad \alpha_i \beta + \beta \alpha_i = 0$

We know: $\mathrm{tr}\,\alpha_i = 0$, $\mathrm{tr}\,\beta = 0$, N - even.

$N=2$

Can take Pauli matrices as α_i, $i=1,2,3$.
Need to find β.
Matrices $\{\sigma_i, \mathbb{1}_{2\times 2}\}$ form a basis $\Rightarrow$ any Hermitian 2×2 matrix can be represented as

$$\beta = a_0 \mathbb{1}_{2\times 2} + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3$$

* exercise: check that conditions

$$\mathrm{tr}\,\beta = 0, \quad \sigma_i\beta + \beta\sigma_i = 0 \Rightarrow \beta \equiv 0,$$

(i.e. $a_0 = a_1 = a_2 = a_3 = 0$).

$\Rightarrow$ cannot find non-trivial β with $N=2$.

Digression: Division algebras over $\mathbb{R}$

$\mathbb{K} = \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ (Hurwitz, 1923)

Division algebra: $z_1, z_2 \in \mathbb{K}$

The eq. $z_1 z_2 = 0 \Rightarrow z_1 = 0$ or (and) $z_2 = 0$

Not „always true": e.g. $\vec{a} \times \vec{b} = 0$ or $\vec{a} \cdot \vec{b} = 0$ allow non-trivial solut.

Quaternions: generalij. of $z = x + iy$ to $q = a + b\hat{i} + cj + dk$

with $i^2 = j^2 = k^2 = -1 \quad ijk = -1$.

($\mathbb{R}^4$ for q as $\mathbb{R}^2$ for $z \in \mathbb{C}$ and $\mathbb{R}^8$ for $\mathbb{O}$.)

Hamilton, 1843 but also Rodrigues and Gauss earlier.

Pauli matrices.

$N=4$

Can check explicitly that all conditions are satisfied with

$$\beta = \begin{pmatrix} \mathbb{1}_{2\times2} & 0 \\ 0 & -\mathbb{1}_{2\times2} \end{pmatrix} \qquad \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

<u>Note</u>: this repres. is <u>irreducible</u>.

Reducible rep. = <u>all</u> matrices are in the form

$$R = \begin{pmatrix} R_1 & 0 & 0 \dots \\ 0 & R_2 & 0 \dots \\ \dots & & \ddots \end{pmatrix}$$

i.e. for $\psi' = R\psi$, $\psi \in V$, the space V splits into invar. subspaces $V = V_1 \oplus V_2 \oplus \dots$ which transform into themselves (do not mix).

<u>How to check if a finite-dim rep. is irreducible?</u>

Schur's lemma (one of them):

If $\{A_1 \ldots\}$ is an irreducible rep. of an algebra (group) in V, $\dim V < \infty$, then the only matrix commuting with <u>all</u> A_i is $M = \lambda \mathbb{1}$, λ = number.

* check that the $N=4$ rep. of Clifford algebra is irreducible.

All $N > 4$ repres. are reducible.

Introduce $\psi^{+} = (\psi_1^{*}, \ldots \psi_4^{*})$.

Multiply (*) by ψ^{+} from the left:

$$i\hbar \psi^{+} \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i} \sum_{k=1}^{3} \psi^{+} \alpha_k \frac{\partial \psi}{\partial x^k} + mc^2 \psi^{+} \beta \psi$$

Now take the eq. Hermitian-conjugate to * and multiply by ψ from the right:

$$-i\hbar \frac{\partial \psi^{+}}{\partial t} \psi = -\frac{\hbar c}{i} \sum_{k=1}^{3} \frac{\partial \psi^{+}}{\partial x^k} \alpha_k \psi + mc^2 \psi^{+} \beta \psi$$

$$\Rightarrow \quad i\hbar \frac{\partial}{\partial t}\left(\psi^{+}\psi\right) = -i\hbar c \sum_{k=1}^{3} \frac{\partial}{\partial x^k}\left(\psi^{+} \alpha_k \psi\right)$$

i.e. $\dfrac{\partial \rho}{\partial t} + \operatorname{div} \vec{j} = 0$,

$$\rho = \psi^{+}\psi, \qquad j^k = c\,\psi^{+} \alpha_k \psi .$$

Covariant form of the Dirac eq. :

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1 \frac{\partial \psi}{\partial x^1} + \alpha_2 \frac{\partial \psi}{\partial x^2} + \alpha_3 \frac{\partial \psi}{\partial x^3}\right) + \beta m c^2 \psi$$

Multiply by β / c, use $\gamma^0 \equiv \beta$, $\gamma^i \equiv \beta \alpha_i$

$$\Rightarrow \qquad i\hbar \gamma^\mu \frac{\partial \psi}{\partial x^\mu} - mc\psi = 0$$

or $(i\hbar \not\partial - mc)\psi = 0$ or $\left(i \not\partial - \frac{mc}{\hbar}\right)\psi = 0$

Note: $\boxed{\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \cdot \mathbb{1}}$

$(\gamma^i)^2 = -\mathbb{1}$, $(\gamma^0)^2 = \mathbb{1}$

With $p^\mu = i\hbar \frac{\partial}{\partial x_\mu}$:

$$(\gamma^\mu p_\mu - mc)\psi = 0$$

or $(\not p - mc)\psi = 0.$

Can take:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

In an external electromagnetic field:

$$p_\mu \to p_\mu - \frac{e}{c} A_\mu$$

$$\Rightarrow \left[\gamma^\mu \left(p_\mu - \frac{e}{c} A_\mu\right) - mc\right]\psi = 0 .$$

Obvious interest: Hydrogen-like atoms

$$e\phi = -Ze^2/r \,,\quad \vec{A} = 0 .$$

The eq. in this case can be solved <u>exactly</u>

(Charles G. Darwin, 1928; W. Gordon, 1928):

$$\mathcal{E}_{n,j} = mc^2 \left[1 + \frac{\alpha^2}{\left(n - j - \frac{1}{2} + \left[(j+\frac{1}{2})^2 - \alpha^2\right]^{\frac{1}{2}}\right)^2} \right]^{-\frac{1}{2}}$$

$$\approx mc^2 \left[1 - \frac{\alpha^2}{2n^2} - \frac{\alpha^4}{2n^4}\left(\frac{n}{j+1/2} - \frac{3}{4}\right) + \dots \right] .$$

$$j = l + s = l \pm 1/2 .$$

The spin of Dirac particle

Consider $S_i = -\frac{i}{4} \varepsilon_{ijk} \alpha_j \alpha_k$

Note: $S_i^{\dagger} = S_i$.

Exercise: show that

1. $[\alpha_i \alpha_k , \alpha_j] = -2 \delta_{ij} \alpha_k + 2 \delta_{kj} \alpha_i$ OK

2. $[S_i , \alpha_j] = -\frac{i}{4} \varepsilon_{ike} [\alpha_k \alpha_e , \alpha_j] =$

$= i \varepsilon_{ije} \alpha_e$ OK

3. $\boxed{[S_i \; S_j] = i \varepsilon_{ijk} S_k}$

For $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$, $\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$

we have $S_i = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$, $\hat{S}_i = \hbar S_i$.

and $\vec{S}^{\,2} = S_1^2 + S_2^2 + S_3^2 = \frac{1}{2}\left(\frac{1}{2}+1\right)\mathbb{1}$.

Operator of spin : $\hat{S}_i = \frac{\hbar}{2}\begin{pmatrix} \sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}$.

Introduce $J_i = l_i + S_i$ $\quad (\hat{\vec{l}} = \hat{\vec{r}} \times \hat{\vec{p}})$

* <u>exercise</u> : for $\hat{H}_D = c\vec{\alpha}\cdot\vec{p} + mc^2\beta$ show

$[l_i, H_D] = -c\alpha_k [p_k\, l_i] = ic\,\varepsilon_{ijk}\alpha_j p_k$

$[S_i, H_D] = c\,p_k [S_i\, \alpha_k] = ic\,\varepsilon_{ijk}\alpha_k p_j$

$\Rightarrow \boxed{[J_i, H_D] = 0}$ (<u>Note</u>: S_i alone and l_i alone do not commute with H_D)

<u>Note</u> : $[J_i\, J_j] = i\,\varepsilon_{ijk} J_k$ (same algebra as l_i and s_i).

We have $[J_z, H_D] = 0$ $\quad [\vec{J}^2, H_D] = 0$

$\Rightarrow$ can take common eigenfunctions

of
$$\begin{cases} H_D \psi_{E,j,M} = E \psi_{E,j,M} \\ J_z \psi_{EjM} = M \psi_{EjM} \\ \vec{J}^2 \psi_{EjM} = j(j+1) \psi_{EjM} \end{cases}$$

Instead of $Y_{lm}(\theta,\varphi)$ will appear spin spherical harmonics $Y_{l,s,j,M}(\theta,\varphi)$.

Helicity

Note that $\hat{S}_i$ do not commute with $\hat{H}_D$.
We can introduce another operator – a projection of $\vec{S}$ on $\vec{p}$: Helicity.
In momentum space,

$$\hat{H}_D = c\,\alpha_i \hat{p}_i + mc^2 \beta$$

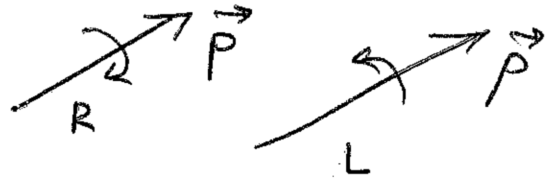
Here $\psi(\vec{p}) = \int e^{-i\vec{p}\vec{r}/\hbar}\, \psi(\vec{r})\, d^3x$

– Fourier repres. of $\psi(\vec{r})$.

Consider operator Λ in moment. space:

$$\Lambda = \frac{\hat{\vec{S}}\,\vec{p}}{|\vec{p}|} = \hat{\vec{S}}\cdot\vec{n}, \quad \text{where } \vec{n} = \frac{\vec{p}}{|\vec{p}|},$$

unit vector along $\vec{p}$.



<u>exercise</u>: show $[\hat{\Lambda}, \hat{H}_D] = 0$.

Note:

$$\Lambda^2 = \frac{1}{4}\begin{pmatrix} (\vec{\sigma}\cdot\vec{n})(\vec{\sigma}\cdot\vec{n}) & 0 \\ 0 & (\vec{\sigma}\vec{n})(\vec{\sigma}\vec{n}) \end{pmatrix}$$

<u>exercise</u>: show that $\sigma_i\sigma_j = \delta_{ij} + i\varepsilon_{ijk}\sigma_k$ for Pauli matrices. Then show that for vectors $\vec{A}, \vec{B}$ the following eq. holds:

$$(\vec{A}\vec{\sigma})(\vec{B}\vec{\sigma}) = A_i B_j \sigma_i \sigma_j = \vec{A}\cdot\vec{B} + i\varepsilon_{ijk} A_i B_j \sigma_k.$$

Therefore,

$$\hat{\Lambda}^2 = \frac{1}{4}\mathbb{1}_{4\times 4}$$

$\Rightarrow$ eigenvalues of $\hat{\Lambda}$ are $\pm 1/2$.

Note:
Helicity of a <u>massive</u> particle is frame-dep.

Transformation properties of Dirac wavefunction ①

10-7-V

Consider Lorentz transf.:

$$\begin{cases} ct' = ct\cosh\eta - x\sinh\eta \\ x' = x\cosh\eta - ct\sinh\eta \\ y' = y \\ z' = z \end{cases}$$

Here $\eta = \operatorname{artanh}\beta = \operatorname{artanh} v/c$:

η = <u>rapidity</u>

Any 4-vector should transform in the same way.

For $j^\mu = (c\rho, \vec{j})$ we have

$$\begin{cases} c\rho' = c\rho\cosh\eta - j_x\sinh\eta \\ j_x' = j_x\cosh\eta - c\rho\sinh\eta \\ j_y' = j_y \\ j_z' = j_z \end{cases}$$

In Dirac's theory, 2

10-7-VI

$$\rho = \psi^{\dagger}\psi = \psi_1^*\psi_1 + \cdots \psi_4^*\psi_4$$

$$j_x = c\,\psi^{\dagger}\alpha_1\psi = c\left(\psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 + \right.$$
$$\left. + \psi_4^*\psi_1\right)$$

Indeed:

$$\alpha_1 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\psi^{\dagger}\alpha_1\psi = (\psi_1^* \cdots \psi_4^*)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} =$$

$$= (\psi_1^* \cdots \psi_4^*)\begin{pmatrix} \psi_4 \\ \psi_3 \\ \psi_2 \\ \psi_1 \end{pmatrix} = \psi_1^*\psi_4 + \psi_2^*\psi_3 + \psi_3^*\psi_2 +$$
$$+ \psi_4^*\psi_1$$

So, $\quad \psi'^{\dagger}\psi' = \psi^{\dagger}\psi \cosh y - \psi^{\dagger}\alpha_1\psi \sinh y$

$$\psi'^{\dagger}\alpha_1\psi' = \psi^{\dagger}\alpha_1\psi \cosh y - \psi^{\dagger}\psi \sinh y$$

$$\psi'^{+}\psi' = \psi^{+}\left(\mathbb{1}\cosh\eta - \alpha_1 \sinh\eta\right)\psi =$$

$$= \psi^{+} e^{-\eta\alpha_1}\psi .$$

<u>Indeed,</u> $e^{-\eta\alpha_1} = \mathbb{1} - \eta\alpha_1 + \frac{\alpha_1^2}{2!}\eta^2 + \frac{\alpha_1\alpha_1^2}{3!}\eta^3$

$$= \mathbb{1}\left(1 + \frac{\eta^2}{2!} + \frac{\eta^4}{4!} + \dots\right) - \alpha_1\left(\eta - \frac{\eta^3}{3!} + \dots\right)$$

$$= \mathbb{1}\cosh\eta - \alpha_1 \sinh\eta . \qquad \left(\alpha_1^2 = \mathbb{1}\right)$$

(In general:

$$e^{-\eta\alpha_1} = \cosh\eta\alpha_1 - \sinh\eta\alpha_1 = \cosh\eta - \alpha_1\sinh\eta).$$

<u>Similarly,</u>

$$\psi'^{+}\alpha_1\psi' = \psi^{+}\left(\alpha_1\cosh\eta - \sinh\eta\right)\psi =$$

$$= \psi^{+}\alpha_1 e^{-\eta\alpha_1}\psi .$$

These eqs can be satisfied with (4)

10-7-VIII

$$\begin{cases} \psi' = e^{-\frac{y}{2}\alpha_1}\psi \\ \psi'^{+} = \psi^{+} e^{-\frac{y}{2}\alpha_1} \end{cases}$$

(Note that: $\alpha_1 e^{-\frac{y}{2}\alpha_1} = e^{-\frac{y}{2}\alpha_1}\alpha_1$,

$\alpha_2 e^{-\frac{y}{2}\alpha_1} = e^{\frac{y}{2}\alpha_1}\alpha_2$.)

Note that ψ transform <u>NOT</u> as vectors (with y) or tensors 2nd rank (with $2y$) but with $y/2$. Such objects are called spinors (tensors of "half-rank").

Now consider rotations. E.g. around z axis on angle φ:

$$\begin{cases} j_x' = j_x \cos\varphi + j_y \sin\varphi \\ j_y' = j_y \cos\varphi - j_x \sin\varphi \\ j_z' = j_z \end{cases}$$

We can check that in this case

10-7-1X

$$\begin{cases} \psi' = e^{i \Sigma_3 \frac{\varphi}{2}} \psi \\ \psi'^{+} = \psi^{+} e^{-i\Sigma_3 \varphi/2} \end{cases} \qquad \Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

ex : show this.

Covariance of the Dirac eq.

$$i\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{mc}{\hbar}\psi(x) = 0 .$$

Consider $x \to x' = \Lambda x$ (Lor. transf.)

The eq. is covar. if

$$i\gamma^\mu \frac{\partial \psi'(x')}{\partial x'^\mu} - \frac{mc}{\hbar}\psi'(x') = 0$$

Assume $\psi'(x') = S(\Lambda)\psi(x)$, find S.

Note $\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu$, $\quad \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial}{\partial x^\nu}$

$$i\Lambda^{-1\nu}{}_{\mu}\gamma^\mu \frac{\partial}{\partial x^\nu} S\psi - \frac{mc}{\hbar} S(\Lambda)\psi(x) = 0$$

i.e. we get $i\gamma^\mu \frac{\partial \psi}{\partial x^\mu} - \frac{mc}{\hbar}\psi = 0$ ⑥

10-7-x

if $S^{-1}(\Lambda)\,\Lambda^{-1\,\nu}{}_{\mu}\,\gamma^\mu S = \gamma^\nu$

or $\boxed{S(\Lambda)\gamma^\nu S^{-1}(\Lambda) = \Lambda^{-1\,\nu}{}_{\mu}\,\gamma^\mu}$.

$S(\Lambda)$ can be found by considering infinitesimal Lor. transf. It has the form

$$S(\Lambda) = e^{-\frac{i}{4}\sigma_{\alpha\beta}\,\omega^{\alpha\beta}},$$

where $\sigma_{\alpha\beta} = \frac{i}{2}[\gamma_\alpha, \gamma_\beta]$, $\omega^{\alpha\beta}$ are parameters of Lor. transf. such as η or φ.

For example, $i\sigma_{01} = \alpha_1$ and

$$S = e^{-\frac{\eta}{2}\alpha_1}$$

for a boost along x direction.

<u>Remark</u>: different bases for γ^μ

10-7-X1

$$i\gamma^\mu \partial_\mu \psi - \frac{mc}{\hbar}\psi = 0$$

Can transform $\psi \to \psi' = U\psi$

$$\gamma^\mu \to \tilde{\gamma}^\mu = U\gamma^\mu U^{-1}$$

with $\det U \neq 0$ => the Dirac eq. will not change. Moreover, the relation

$$\{\gamma^\mu \gamma^\nu\} = 2\eta^{\mu\nu}$$

remains unchanged. We want $\tilde{\gamma}^\mu$ to remain <u>Hermitian</u> => U is unitary.

<u>Indeed</u>: $\tilde{\gamma}^{\mu\dagger} = \tilde{\gamma}^\mu$ implies

$$\tilde{\gamma}^{\mu\dagger} = \underline{(U^{-1})^\dagger \gamma^{\mu\dagger} U^\dagger} = \tilde{\gamma}^\mu = \underline{U\gamma^\mu U^{-1}}$$

$$\Rightarrow U^\dagger = U^{-1} \quad (\text{since } \gamma^{\mu\dagger} = \gamma^\mu)$$

or $UU^\dagger = \mathbb{1}$ (unitary matrices).

Using this freedom, can get convenient representations for γ matrices and different „types" of spinors ψ (e.g. real ones — called Majorana spinors or the 2 two-component pairs transforming separately under Lor. transform.: they realise $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ repres. of Lor. group - need ~~both~~ to include space inversions).

Non-relativistic limit

Stationary solutions $\psi(\vec{r},t) = e^{-i\varepsilon t/\hbar}\psi(\vec{r})$

$$\Rightarrow \quad \hat{H}\psi(\vec{r}) = \varepsilon\psi(\vec{r})$$

Let $\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$, $\quad \varphi \equiv \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$,

$\chi = \begin{pmatrix}\psi_3\\ \psi_4\end{pmatrix}$.

Then:
$$\begin{cases} \varepsilon\varphi = c\,\vec{\sigma}\cdot\hat{\vec{p}}\,\chi + mc^2\varphi \\ \varepsilon\chi = c\,\vec{\sigma}\cdot\hat{\vec{p}}\,\varphi - mc^2\chi \end{cases}$$

i.e. $\quad A\begin{pmatrix}\varphi\\ \chi\end{pmatrix} = 0$

$$A = \begin{pmatrix} mc^2 - \varepsilon & c\,\vec{\sigma}\cdot\hat{\vec{p}} \\ c\,\vec{\sigma}\cdot\hat{\vec{p}} & -(mc^2+\varepsilon) \end{pmatrix}$$

$\det A = 0 \Leftrightarrow$ nontrivial solutions

$\Rightarrow \varepsilon = \pm\sqrt{p^2c^2+m^2c^4}$

(use $(\vec{\sigma}\vec{A})(\vec{\sigma}\vec{B}) = \vec{A}\cdot\vec{B} + i\vec{\sigma}\cdot(\vec{A}\times\vec{B})$ to show this).

Now, $\qquad \chi = \dfrac{c\,\vec{\sigma}\cdot\vec{\hat{p}}}{mc^2+\varepsilon}\,\varphi$

If in a given ref. frame the motion is non-relativ., $\quad \varepsilon = \pm(mc^2+E)$, $\quad E \ll mc^2$.

For sol. with $\varepsilon > 0$

$$|\chi| = \left|\frac{c\,\vec{\sigma}\vec{\hat{p}}\,\varphi}{mc^2+\varepsilon}\right| \sim \left|\frac{p}{2mc}\,\varphi\right| \ll |\varphi|$$

For sol. with $\varepsilon < 0$

$$|\chi| = \left|\frac{c\,\vec{\sigma}\cdot\vec{\hat{p}}}{mc^2-|\varepsilon|}\,\varphi\right| \sim \left|\frac{2mc}{p}\,\varphi\right| \gg |\varphi|$$

$\Rightarrow$ in non-rel. case, 2 out of 4 components are small.

Stationary solutions in external field $A^\mu = (\Phi, \vec{A})$:

$$(\varepsilon - e\Phi - mc^2)\varphi = c\vec{\sigma}(\hat{\vec{p}} - \tfrac{e}{c}\vec{A})\chi$$

$$(\varepsilon - e\Phi + mc^2)\chi = c\vec{\sigma}(\vec{p} - \tfrac{e}{c}\vec{A})\varphi$$

With $\varepsilon = mc^2 + E$:

$$(E - e\Phi)\varphi = c\vec{\sigma}(\vec{p} - \tfrac{e}{c}\vec{A})\chi$$

$$(E - e\Phi + 2mc^2)\chi = c\vec{\sigma}(\vec{p} - \tfrac{e}{c}\vec{A})\varphi$$

For $|E| \ll mc^2$, $|e\Phi| \ll mc^2$, the sec. eq:

$$\chi = \frac{\vec{\sigma}(\vec{p} - \frac{e}{c}\vec{A})}{2mc}\varphi + \dots \quad \rightarrow \text{into first}$$

$$\Rightarrow (E - e\Phi)\varphi = \frac{1}{2m}\left[\vec{\sigma}\cdot(\vec{p} - \tfrac{e}{c}\vec{A})\vec{\sigma}\cdot(\vec{p} - \tfrac{e}{c}\vec{A})\right]\varphi$$

Use the identity above to show

$$(E - e\Phi)\varphi = \frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m}\varphi - \frac{e\hbar}{2mc}\vec{\sigma}\cdot\vec{B}\cdot\varphi,$$

$\vec{B} = \mathrm{curl}\,\vec{A}$.

For non-rel. spinor $\varphi \sim e^{-iEt/\hbar}$:

$$i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{(\vec{p} - \frac{e}{c}\vec{A})^2}{2m} + e\Phi - \mu_0\,\vec{\sigma}\cdot\vec{B}\right]\varphi,$$

where $\mu_0 = \frac{e\hbar}{2mc}$. (This is Pauli eq.).

- Dirac eq (with min. coupling to electromagnetic field) gives $\mu_0 = \frac{e\hbar}{2mc}$
- The value of μ_0 is (almost) correct for the electron but for other particles with spin $1/2$ corrections due to interacting with other fields are significant.
- For electron, expect corrections due

to QED (Lamb shift, self-inter. etc)

$\Rightarrow$ can be computed in QED, agrees with experiment.

- Non-minimal coupling can change the value of μ_0. For example, the term $\lambda\beta[\gamma^\mu, \gamma^\nu] F_{\mu\nu} \psi$ can be added to $\hat{H}_D$.* This term is compatible with all symmetries. In this sense, Dirac theory does not fix the value of μ_0. This is further clarified in Quantum Field Theory.

*Pointed out by W. Pauli.