Advanced QM

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- Landau, LD
  Lifshitz, EM „QM" Pergamon Press (1977)

- Sakurai JJ: „Modern QM"
  Addison-Wesley

- Bjorken, Drell:
  Relativistic QM, McGraw Hill

- Wu, Ohmura: Quantum theory of scattering
  Prentice Hall, 1962

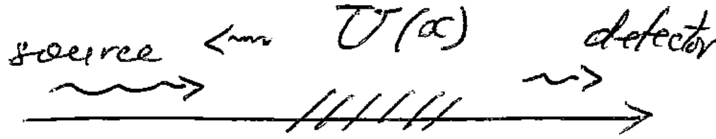
- J. Taylor: Scattering theory: Dover

Aspects of 1-dim QM scattering

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U(x)\psi$$

$$\hat{H} = \frac{\hat{p}^2}{2m} + U(x), \quad \psi = \psi(x,t)$$

Non-rel. 1d scattering:



Stationary states of {continuous / discrete} spectrum in 1d

$$\psi(x,t) = e^{-iEt/\hbar}\,\psi(x) \;:\; -\frac{\hbar^2}{2m}\psi'' + U(x)\psi = E\psi$$

$$k^2 = 2mE/\hbar^2 \quad \boxed{\psi'' - u(x)\psi = -k^2\psi}$$

→ source of particles of energy E at $x \to -\infty$
probability of their detection at $x \to +\infty$

Transmission coefficient (transmissivity) $T(E)$:
the prob. that a particle emitted with energy E at $x \to -\infty$ will go through to the right ($x \to +\infty$).

$$\psi_- \sim e^{ik_- x} + A e^{-ik_- x} \qquad x \to -\infty$$

$$\psi_+ \sim S e^{ik_+ x} \qquad x \to +\infty$$

$$k_\pm = \sqrt{2m(E - U_\pm)}/\hbar, \quad U_\pm = U(\pm\infty).$$

$$T(E) = \frac{k_+}{k_-} |S(E)|^2$$

– ~~отношение плотностей потоков вероятности~~

Probability current density ratio of ~~incident and~~ reflected and incident waves:

$$\vec{j}(\psi) = \frac{i\hbar}{2m}\left(\psi \nabla \psi^* - \psi^* \nabla \psi\right) \quad (*)$$

$$\frac{\partial |\psi|^2}{\partial t} + \operatorname{div} \vec{j} = 0$$

$$j_-^{inc} = \frac{i\hbar}{2m}\left(-ik_- - ik_-\right) = \frac{\hbar k_-}{m}$$

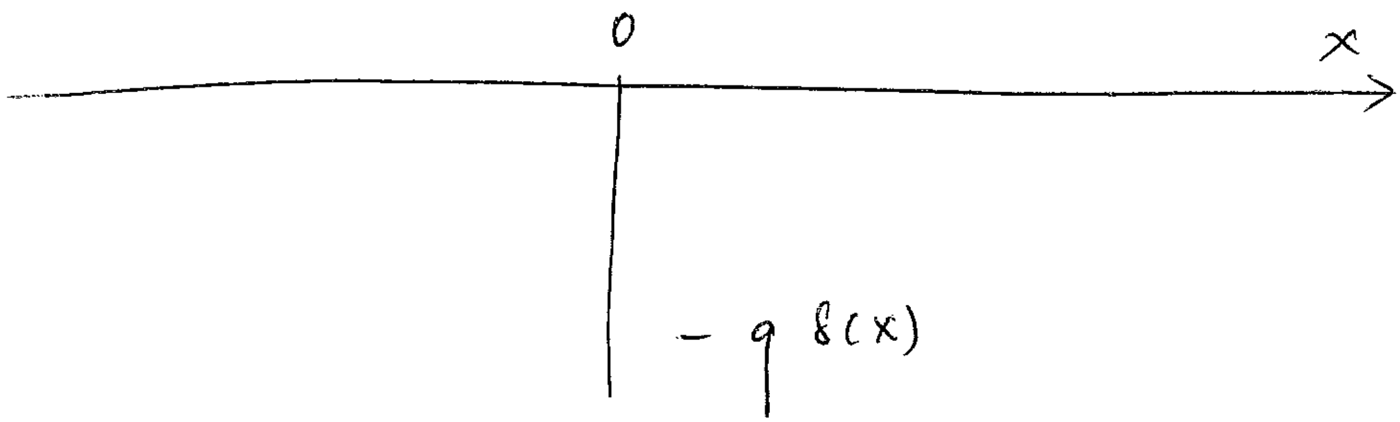
$$j_+^{tr} = \frac{\hbar k_+}{m} |S(E)|^2$$

$$T(E) = \frac{j_+^{tr}}{j_-^{inc}} = \frac{k_+}{k_-} |S(E)|^2.$$

<u>Note</u>: $$R(E) = \frac{j_-^{ref}}{j_-^{inc}} = |A|^2$$

<u>ex</u>: show $T(E) + R(E) = 1$ for Hermitian $\hat{H}$ (for real $U(x)$ in this case).

<u>Example 1</u>: $U(x) = -q\,\delta(x)$.



$$\psi = \psi_- = e^{ikx} + A e^{-ikx} \quad \text{for } x < 0$$

$$\psi = \psi_+ = S e^{ikx} \quad \text{for } x > 0$$

$$-\frac{\hbar^2}{2m}\psi'' - q\,\delta(x)\,\psi = E\psi$$

$$\psi'' + \frac{2mq}{\hbar^2}\delta(x)\,\psi = -k^2\psi$$

Conditions on ψ: contin., ψ' - disc., so $\psi'' \sim \delta(x)$

$$\int_{-\varepsilon}^{\varepsilon}\psi''\,dx + \frac{2mq}{\hbar^2}\int_{-\varepsilon}^{\varepsilon}\delta(x)\,\psi(x)\,dx = -k^2\int_{-\varepsilon}^{\varepsilon}\psi\,dx$$

$$\psi'(\varepsilon) - \psi'(-\varepsilon) + \frac{2mq}{\hbar^2}\psi(0) = 0$$

$$\Rightarrow \begin{cases} \psi(-\varepsilon) = \psi(\varepsilon) \\ \psi'(-\varepsilon) - \psi'(\varepsilon) = \dfrac{2mq}{\hbar^2}\psi(0) \end{cases}$$

$$\Rightarrow \begin{cases} 1 + A = S \\ ik - Aik = ikS = \frac{2mq}{\hbar^2}\psi(0) = \frac{2mq}{\hbar^2} S \end{cases}$$

$$1 + A = S$$

$$1 - A = S\left(1 + \frac{2mq}{\hbar^2 ik}\right)$$

$$2 = 2\left(1 + \frac{mq}{ik\hbar^2}\right) S \quad \Rightarrow \quad S = \frac{ik}{ik + \frac{mq}{\hbar^2}}$$

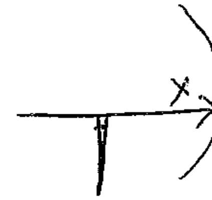
$$2A = -\frac{2mq}{\hbar^2 ik} S$$

$$A = -\frac{mq}{\hbar^2 ik}\,\frac{ik}{ik + \frac{mq}{\hbar^2}} = -\frac{\varkappa}{ik + \varkappa} \qquad \left(\varkappa \equiv \frac{mq}{\hbar^2}\right)$$

$$T = |S|^2 = \frac{-ik}{-ik + \varkappa}\,\frac{ik}{ik + \varkappa} = \frac{k^2}{k^2 + \varkappa^2}.$$

$$R = |A|^2 = \frac{\varkappa^2}{k^2 + \varkappa^2}. \qquad (\text{Note } R + T = 1)$$

$$S(E) = \frac{ik}{ik + \varkappa}; \qquad \left(E_0 = -\frac{mq^2}{2\hbar^2} \text{ in } \right)$$



$$ik = -\varkappa \Rightarrow k = i\varkappa \text{ or } k^2 = -\varkappa^2 = 2mE/\hbar^2 : E = -\frac{\hbar^2}{2m}\varkappa^2$$

$$T(E) = \frac{k^2(E)}{k^2(E) + \varkappa^2}$$

$k_* = \pm i\varkappa$ (poles)

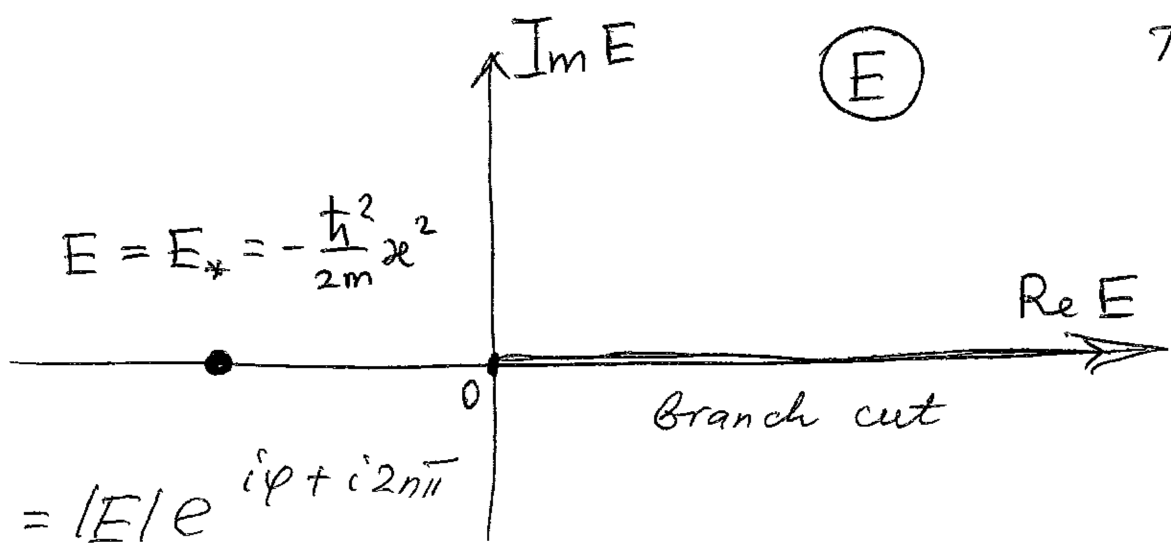
Recall $k^2 = 2mE/\hbar^2 \Rightarrow E_* = \frac{\hbar^2}{2m} k^2 = -\frac{\hbar^2}{2m}\varkappa^2$

$= -\frac{mg^2}{2\hbar^2} = E_0$ (energy of a bound state

in $U(x) = -g\,\delta(x)$).

$$S(E) = \frac{ik}{ik + \varkappa}, \quad k = \sqrt{2mE}/\hbar$$

$ik = -\varkappa$, $k = i\varkappa$ pole $\varkappa = \frac{mg}{\hbar^2} > 0$ for a well



$E = |E| e^{i\varphi + i2n\pi}$

$\sqrt{E} = |E|^{1/2} e^{\frac{i\varphi}{2} + in\pi}$ $n = 0, 1$

$n=0$ sheet: $\sqrt{E} = |E|^{1/2} e^{i\varphi/2}$

$\Rightarrow E<0 \; (\varphi = \pi) \Rightarrow \sqrt{E} = i|E|^{1/2}$

$n=1$ sheet: $\sqrt{E} = |E|^{1/2} e^{i\varphi/2 + i\pi}$

$\Rightarrow \sqrt{E} = |E|^{1/2} e^{i3\pi/2} = -i|E|^{1/2} \quad (E<0)$

So, $\quad k = i\varkappa$:

1) $\varkappa > 0$: $\quad \frac{\sqrt{2m}}{\hbar}|E|^{1/2} i = i\varkappa > 0 \quad$ OK

(pole is on the $n=0$ sheet)
"physical"

$U(x) = -\varkappa\,\delta(x)$ has a bound state

2) $\varkappa < 0$: $\quad -i\frac{\sqrt{2m}\,|E|^{1/2}}{\hbar} = -i|\varkappa| \quad$ OK

(pole is on the $n=1$ sheet)

$U(x) = |\varkappa|\,\delta(x)$ has no bound states.

$$T(E) = \frac{k^2}{k^2+\varkappa^2} = \frac{E}{E+|E_*|}, \quad E_* = -\frac{\hbar^2}{2m}\varkappa^2$$

Observation:

poles of $S(E)$ on the negative real axis

$\Longleftrightarrow$ bound states of $U(x)$

Remark: we expect $S(E)$ to be an analytic function of E in the upper half-plane (of complex E). Recall that $\psi(t,x) \sim e^{-iEt/\hbar}$ and $E = \operatorname{Re} E + i \operatorname{Im} E$ corresponds to $\psi \sim e^{\operatorname{Im} E\, t/\hbar}$. For $\operatorname{Im} E > 0$ this increases without bound for $t \to \infty$ $\Rightarrow$ violates prob. conservation $w \sim |\psi|^2$. We can have $\operatorname{Im} E \leq 0$ as $|\psi|^2 \to 0$ in this particular channel means $|\psi|^2$ increases in other channels so that ~~total~~ $w = 1$.

Resonances

The pot. $U = -\alpha\,\delta(x)$ is too simple - it has zero size. Improve this by considering $U(x) = -\frac{g}{2}\left[\delta(x+a) + \delta(x-a)\right]$

or a pot. well. (8)