Sample Exam for Third Year Course VI

Condensed Matter Physics (AKA Solid State Physics)

Hilary Term 2011

Each problem is 25 points

1. Explain the meaning of the terms *lattice* and *basis* when used to describe a crystal structure. [4]

The crystal structure of fluorite (CaF_2) has a face-centred cubic lattice and a basis consisting of Ca ions at $(0, 0, 0)$ and F ions at $\pm(\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$ referred to the conventional cubic unit cell. Make a sketch of the cubic unit cell of CaF_2 projected down the z axis onto the $z = 0$ plane. Indicate the z coordinate next to each ion. Identify on separate diagrams the sets of planes with Miller indices (1 1 0), (2 0 0) and (4 0 0). [8]

The crystal structure of Ca metal also has a face-centred cubic lattice. A polycrystalline specimen of Ca is examined by X-ray diffraction. The wavelength of the X-rays is 0·200 nm. A Bragg peak corresponding to the (2 0 0) plane is observed at an angle of 42·0° from the undeflected beam. Calculate the size of the cubic unit cell of Ca, and explain why no Bragg scattering is observed from the (1 1 0) plane. [9]

Explain qualitatively how the ratio of the (2 0 0) and (4 0 0) Bragg peak intensities in the X-ray powder diffraction patterns of Ca and of CaF_2 would differ. [4]

2. Explain the meaning of the term *phonon*. Describe briefly how the phonon dispersion curves in a crystal can be measured. [6]

Consider a one-dimensional, monatomic lattice of period a containing atoms of mass M. The atoms have long-range interatomic forces obeying Hooke's law. The force on atom p caused by atom $p+n$ is proportional to the difference of their longitudinal displacements; the force constant C_n between them depends on n but not p. Obtain the equation of motion of atom p. Show that the dispersion relation of lattice vibrations is

$$\omega(k) = \left(\frac{4}{M}\right)^{\frac{1}{2}} \left(\sum_{n>0} C_n \, \sin^2 \frac{nka}{2}\right)^{\frac{1}{2}} .$$

Hence show that

$$C_n = -\left(\frac{Ma}{2\pi}\right) \int_{-\pi/a}^{\pi/a} \omega^2(k) \cos nka \, \mathrm{d}k.$$

[10]

The value of C_n falls off rapidly with n. Assuming $nka \ll 1$, show that the dispersion relation $\omega(k)$ is approximately linear in k. Hence show that the speed of sound u in the long-wavelength limit is given by

$$u = \left(\frac{a^2}{M}\right)^{\frac{1}{2}} \left(\sum_{n>0} n^2 \, C_n\right)^{\frac{1}{2}} .$$

[4]

Sketch the dispersion relation for all k, including only nearest-neighbour interactions (i.e. $C_n = 0$ for $|n| > 1$). Discuss modifications to the shape of this dispersion curve due to the inclusion of long-range interatomic forces. [5]

3. Explain the terms *Fermi energy* E_{F} and *Fermi temperature* T_{F}. [6]

Find a formula for $g(E)$, the density of states per unit energy of a two-dimensional gas of free electrons of mass m. The number N of electrons is given by

$$N = \int_0^\infty \frac{g(E)\,\mathrm{d}E}{\mathrm{e}^{(E-\mu)/k_{\mathrm{B}}T} + 1} .$$

Using the substitution $x = \mathrm{e}^{(E-\mu)/k_{\mathrm{B}}T} + 1$ or otherwise, show that the chemical potential μ is given by

$$\mu = k_{\mathrm{B}}T \ln\{\exp(n\pi\hbar^2/mk_{\mathrm{B}}T) - 1\},$$

where n is the number of electrons per unit area. [13]

Show that the Fermi energy is given by

$$E_{\mathrm{F}} = \frac{n\pi\hbar^2}{m} .$$

Explain why, for a typical three-dimensional metal, the Fermi–Dirac formula may for some purposes be approximated by

$$\frac{1}{\mathrm{e}^{(E-E_{\mathrm{F}})/k_{\mathrm{B}}T} + 1} .$$ [6]

4. Explain briefly the origin of the electronic band gap in a typical electrical insulator. [6]

The periodic potential $V(x)$ experienced by an electron in a one-dimensional crystal may be given in the form

$$V(x) = V_0 + V_G\, e^{-iGx} + V_{-G}\, e^{+iGx},$$

where G is the reciprocal lattice vector, and $|V_G| = |V_{-G}|$. Explain why a suitable wavefunction for an electron in such a potential may be written to a first approximation as

$$\psi(x) = Ae^{ikx} + Be^{i(k-G)x}.$$ [3]

assuming *k* is near *G/2*.

By substituting $\psi(x)$ into the Schrödinger equation and comparing coefficients in e^{ikx} and $e^{i(k-G)x}$, show that the energy of an electron of mass m and wavevector k at the zone boundary is given by

$$E = V_0 + \frac{\hbar^2 k^2}{2m} \pm |V_G|\,.$$

Discuss the significance of each of the three terms on the right-hand side of this equation in terms of band theory. [10]

Using this result explain why diamond is a good electrical insulator, whereas silicon and germanium, which have the same structure type as diamond, are semiconductors. (In the diamond structure there are two tetravalent atoms in the basis.) [6]