Exercises for San Sebastian Summer School

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No school is complete without homework assignments! You can do these on the beach, or at night after a patxaran. Or do them when you get home. Either way, I promise you will learn more if you do them!

Problem 1 Quantum Hall Conductivity vs Conductance

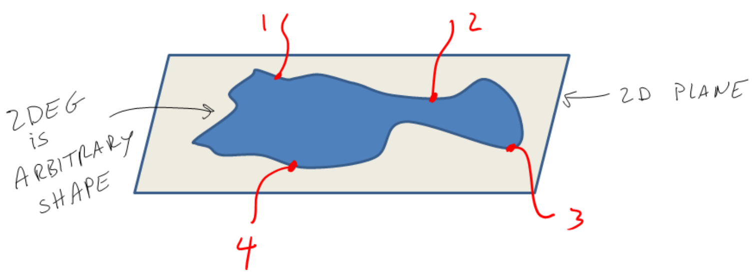

Figure 1: A 2D electron gas of arbitrary shape with contacts 1,2,3,4 attached on its perimeter in clockwise order

Consider a two dimensional electron gas (2DEG) of arbitrary shape in the plane with four contacts (1,2,3,4) attached at its perimeter in a clockwise order as shown in Fig. 1. The conductivity tensor σ_{ij} relates the electric field to the current via

$$j_i = \sigma_{ij} E_j \tag{1}$$

where indices i and j take values $\hat{x}$ and $\hat{y}$ (and sum over j is implied). Assume that this is a quantized hall system with quantized hall conductance s. In other words, assume that

$$\sigma = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix} \tag{2}$$

Show that the following two statements are true independent of the shape of the sample.

(a) Suppose current I is run from contact 1 to contact 2, show that the voltage measured between contact 3 and 4 is zero.

(b) Suppose current I is run from contact 1 to contact 3, show that the voltage measured between contact 2 and 4 is $V = I/s$.

Note: The physical measurements proposed here measure the *conductance* of the sample, the microscopic quantity σ is the *conductivity*.

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Problem 2 *Two Terminal Conductance*

Consider a two-terminal hall bar with one Landau level filled, with one side of the bar at μ_L, T_L and the other side at μ_R, T_R. We showed in lecture that (with $T_L = T_R = T = 0$) the conductance from right to left is $J = G(\mu_R - \mu_L)$ with $G = e^2/h$. The thermal current can be written as

$$j^q = \int \frac{dk}{2\pi}(\epsilon_k - \mu)v_k n_F(\beta(\epsilon_k - \mu))$$

The thermal conductance of the sample is $J^Q = K(T_R - T_L)$ where $\mu_L = \mu_R$. Show

$$\frac{K}{TG} = \frac{\pi^2 k_B^2}{3e^2}$$

You may find the following integral useful

$$\int_{-\infty}^{\infty} dx x^2 \frac{d}{dx}\frac{1}{e^x + 1} = -\frac{\pi^2}{3}$$

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Problem 3 *Counting Edge Modes*

Consider an integer quantum Hall edge filled up to the chemical potential. The ground state we write as

$$\ldots \mathbf{111111} \,|\, 0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ldots$$

where **1** marks a filled orbital and 0 marks an empty orbital. The vertical line marks the chemical potential of the system (the highest momentum occupied orbital in the ground state). To create an edge excitatation we must promote a fermion to a higher orbital, i.e, we move some fermion further right. For example, if we want a state with one additional unit of angular momentum we want to promote a fermion to the the right by one more s tep. There is only one way to do this without violating the Pauli exculusion rule,highest angular momentum occupied orbital in the ground state)

$$(\Delta k = 1) \qquad \ldots \mathbf{11111}\,0 \,|\, \mathbf{1}\,0\,0\,0\,0\,0\,0\,0\,0\,0 \ldots$$

Show that the number of ways to create an excitation of $\Delta k = M$ steps is equal to the number of different ways to partition the integer M.

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Problem 4 *About the Lowest Landau Level*

If you have never before actually solved the problem of an electron in two dimensions in a magnetic field, it is worth doing. Even if you have done it before, it is worth doing again.

Consider a two dimensional plane with a perpendicular magnetic field $\vec{B}$. Work in symmetric gauge $\vec{A} = \frac{1}{2}\vec{r} \times \vec{B}$.

(a)(This is the hard part, see below for hints if you need them.) Show that the single electron Hamiltonian can be rewritten as

$$H = \hbar\omega_c(a^\dagger a + \frac{1}{2}) \tag{3}$$

where $\omega_c = eB/m$ and

$$a = \sqrt{2}\ell\left(\bar{\partial} + \frac{1}{4\ell^2}z\right) \tag{4}$$

with $z = x+iy$ and $\bar{\partial} = \partial/\partial\bar{z}$ with the overbar meaning complex conjugation. Here ℓ is the magnetic length $\ell = \sqrt{\hbar/eB}$.

(b) Confirm that

$$[a, a^\dagger] = 1 \tag{5}$$

and therefore that the energy spectrum is that of the harmonic oscillator

$$E_n = \hbar\omega_c(n + \frac{1}{2}) \tag{6}$$

(c) Once you obtain Eq. 3, show that any wavefunction

$$\psi = f(z)e^{-|z|^2/4\ell^2} \tag{7}$$

with f any analytic function is an eigenstate with energy $E_0 = \frac{1}{2}\hbar\omega_c$. Show that an orthogonal basis of wavefunctions in the lowest Landau level (i.e., with eigenenergy E_0) is given by

$$\psi_m = N_m z^m e^{-|z|^2/4\ell^2} \tag{8}$$

where N_m is a normalization constant. Show that the maximum amplitude of the wavefunction ψ_m is a ring of radius $|z| = \ell\sqrt{2m}$ and calculate roughly how the amplitude of the wavefunction decays as the radius is changed away from this value.

(d) Defining further

$$b = \sqrt{2}\ell\left(\partial + \frac{1}{4\ell^2}\bar{z}\right) \tag{9}$$

with $\partial = \partial/\partial z$, Show that the operator b also has canonical commutations

$$[b, b^\dagger] = 1 \tag{10}$$

but both b and $b^\dagger$ commute with a and $a^\dagger$. Conclude that applying b or $b^\dagger$ to a wavefunction does not change the energy of the wavefunction.

(e) show that the $\hat{z}$ component of angular momentum (angular momentum perpendicular to the plane) is given by

$$L = \hat{z}\cdot(\vec{r}\times\vec{p}) = \hbar(b^\dagger b \ - a^\dagger a \ ) \tag{11}$$

Conclude that applying b or $b^\dagger$ to a wavefunction changes its angular momentum, but not its energy.

(f) [Harder] Let us write an arbitrary wavefunction (not necessarily lowest Landau level) as a polynomial in z and $\bar{z}$, times the usual gaussian factor. Show that projection of this wavefunction to the lowest Landau level can be performed by moving all of the $\bar{z}$ factors all the way to the left and replacing each $\bar{z}$ with $2\ell^2\partial_z$.

Hints to part a: First, define the antisymmetric tensor ϵ_{ij}, so that the vector potential may be written as $A_i = \frac{1}{2}B\epsilon_{ij}r_j$. We have variables p_i and r_i that have canonical commutations (four scalar variables total). It is useful to work with a new basis of variables. Consider the coordinates

$$\begin{aligned} \pi_i^{(\alpha)} &= p_i + \alpha\frac{\hbar}{2\ell^2}\epsilon_{ij}r_i & (12)\\ &= \frac{\hbar}{\ell^2}\epsilon_{ij}\xi_j & (13)\end{aligned}$$

defined for $\alpha = \pm 1$. Here $\alpha = +1$ gives the canonical momentum. Show that

$$\left[\pi_i^{(\alpha)}, \pi_j^{(\beta)}\right] = i\alpha\epsilon_{ij}\delta_{\alpha\beta}\frac{\hbar^2}{\ell^2} \tag{14}$$

The Hamiltonian

$$H = \frac{1}{2m}(p_i + eA_i)(p_i + eA_i) \tag{15}$$

can then be rewritten as

$$H = \frac{1}{2m}\pi_i^{(+1)}\pi_i^{(+1)} \tag{16}$$

with a sum on $i = \hat{x}, \hat{y}$ implied. Finally use

$$\begin{aligned} a &= (-\pi_y^{(+1)} + i\pi_x^{(+1)})\frac{\ell}{\sqrt{2}\hbar} && (17) \\ b &= (\pi_y^{(-1)} + i\pi_x^{(-1)})\frac{\ell}{\sqrt{2}\hbar} && (18) \end{aligned}$$

to confirm that a and b are given by Eqs. 4 and 9 respectively. Finally confirm Eq. 3 by rewriting Eq. 16 using Eqs. 17 and 18.

A typical Place to get confused is the definition of ∂. Note that

$$\begin{aligned} \partial z &= \bar{\partial}\bar{z} = 1 && (19) \\ \bar{\partial} z &= \partial \bar{z} = 0 && (20) \end{aligned}$$

Hints to part f: Rewrite the operators $a, a^\dagger, b, b^\dagger$ such that they operate on polynomials, but not on the Gaussian factor. Construct $\bar{z}$ in terms of these operators. Then project.

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Problem 5 *Filled Lowest Landau Level*

Show that the filled Lowest Landau level of non-interacting electrons (a single slater determinant) can be written as

$$\Psi_m^0 = \mathcal{N} \prod_{1 \le i < j \le N} (z_i - z_j)^1 \prod_{1 \le i \le N} e^{-|z|^2/4\ell^2} \tag{21}$$

with $\mathcal{N}$ some normalization constant. I.e, this is the Laughlin wavefunction with exponent $m = 1$.

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Problem 6 *Laughlin Plasma Analogy*

Consider the Laughlin wavefunction for N electrons at positions z_i

$$\Psi_m^0 = \mathcal{N} \prod_{1 \le i < j \le N} (z_i - z_j)^m \prod_{1 \le i \le N} e^{-|z|^2/4\ell^2} \tag{22}$$

with $\mathcal{N}$ a normalization constant. The probability of finding particles at positions $\{z_1, \ldots, z_N\}$ is given by $|\Psi_m(z_1, \ldots z_N)|^2$.

Consider now N classical particles at temperature $\beta = \frac{1}{k_b T}$ in a plane interacting with logarithmic interactions $v(\vec{r}_i - \vec{r}_j)$ such that

$$\beta v(\vec{r}_i - \vec{r}_j) = -2m \log(|\vec{r}_i - \vec{r}_j|) \tag{23}$$

in the presence of a background potential u such that

$$\beta u(|\vec{r}|) = |\vec{r}|^2/(2\ell^2) \tag{24}$$

Note that this log interaction is "Coulombic" in 2d (i.e., $\nabla^2 v(\vec{r}) \propto \delta(\vec{r})$).

(a) Show that the probability that these classical particles will take positions $\{\vec{r}_1, \ldots, \vec{r}_N\}$ is given by $|\Psi_m^0(z_1, \ldots z_N)|^2$ where $z_j = x_j + iy_j$ is the complex representation of position $\vec{r}_i$. Argue that the mean particle density is constant up to a radius of roughly $\ell\sqrt{Nm}$. (Hint: Note that u is a neutralizing background. What configuration of charge would fully screen this background?)

(b) Now consider the same Laughlin wavefunction, but now with M quasiholes inserted at positions $w_1, \ldots, w_M$.

$$\Psi_m = \mathcal{N}(w_1, \ldots, w_M) \left[\prod_{1 \le i \le N} \prod_{1 \le \alpha \le M} (z_i - w_\alpha) \right] \Psi_m^0 \tag{25}$$

where $\mathcal{N}$ is a normalization constant which may now depend on the positions of the quasiholes. Using the plasma analogy, show that the $w - z$ factor may be obtained by adding additional logarithmically interacting charges at positions w_i,with $1/m$ of the charge of each of the z particles

(c) Note that in this wavefunction the z's are physical parameters (and the wavefunction must be single-valued in z's), but the w's are just parameters of the wavefunction – and so the function $\mathcal{N}$ could be arbitrary — and is only fixed by normalization. Argue using the plasma analogy that in order for the wavefunction to remain normalized (with respect to integration over the z's) as the w's are varied, we must have

$$|\mathcal{N}(w_1, \ldots, w_M)| = \mathcal{K} \prod_{1 \le \alpha < \gamma \le M} |w_\alpha - w_\gamma|^{1/m} \prod_{1 \le \alpha \le M} e^{-|w_\alpha|^2/(4m\ell^2)} \tag{26}$$

with $\mathcal{K}$ a constant so long as the $w's$ are not too close to each other. (Hint: a plasma will screen a charge).

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