# Exercises on Landau Theory of Phase Transitions (MMathPhys and MPhysC6)

**Question 1.** Consider a Landau expansion of the free energy of the form

$$F = \frac{a}{2}m^2 + \frac{b}{4}m^4 + \frac{c}{6}m^6$$

with $c > 0$. Examine the phase diagram in the $a-b$ plane, and show that there is a line of critical transitions $a = 0$, $b > 0$ which joins a line of first order transitions $b = -4(ca/3)^{1/2}$ at a point $a = b = 0$ known as a tricritical point.

Supposing that $a$ varies linearly with temperature and that $b$ is independent of temperature, compare the value of the exponent $\beta$ at the tricritical point with its value on the critical line.

From Yeomans, *Statistical Mechanics of Phase Transitions*

**Question 2.**
(a) Discuss how an order parameter may be used to characterise symmetry breaking at a phase transition.
(b) Argue that the uniaxial ferromagnet-paramagnet transition can be described by by a Landau free energy of the form

$$F = \int d^3\mathbf{r}\left[\frac{1}{2}|\nabla\phi(\mathbf{r})|^2 - h\phi(\mathbf{r}) + \alpha_2\phi^2(\mathbf{r}) + \alpha_3\phi^3(\mathbf{r}) + \alpha_4\phi^4(\mathbf{r})\right].$$

What can you say about $\alpha_4$?
(c) What is the nature of the transition for $h = 0$ if $\alpha_3 \neq 0$? Explain your answer.
(d) Now assume that $\alpha_3 = h = 0$. Argue that close to the critical point

$$\alpha_2 = At\ , \quad t = \frac{T - T_c}{T_c} \text{ and } A > 0.$$

(e) Derive the equation characterizing the saddle point solution for $\alpha_3 = h = 0$. What are the configurations $\phi$ with the lowest free energy for $h = 0$, at $T > T_c$ and at $T < T_c$? Why are these $\mathbf{r}$ independent?
(f) Now consider more general solutions to the saddle point equation in the low-temperature phase. With suitable boundary conditions the saddle point solutions for the order parameter are functions of $x$ only, i.e. $\phi = \phi(x)$. Show that in this case

$$E = \frac{1}{2}\left[\frac{d\phi(x)}{dx}\right]^2 - \alpha_2\phi^2 - \alpha_4\phi^4$$

is independent of $x$. Construct a solution $\phi(x)$ such that

$$\lim_{x\to\infty}\phi(x) = \phi_1\ , \quad \lim_{x\to-\infty}\phi(x) = \phi_2,$$

where $\phi_{1,2}$ are the solutions found in (d). Hint: determine $E$ for such solutions first.

**Question 3.** A system with a real, two-component order parameter $(\phi_1(\mathbf{r}), \phi_2(\mathbf{r}))$ has a free energy

$$F = \int \mathrm{d}^d\mathbf{r}\left[\frac{1}{2}|\nabla\phi_1(\mathbf{r})|^2 + \frac{1}{2}|\nabla\phi_2(\mathbf{r})|^2 - \frac{1}{2}\left(\phi_1^2(\mathbf{r}) + \phi_2^2(\mathbf{r})\right) + \frac{1}{4}\left(\phi_1^2(\mathbf{r}) + \phi_2^2(\mathbf{r})\right)^2\right].$$

Find the order-parameter values $\Phi_1, \Phi_2$ that minimise this free energy. Consider small fluctuations around such state, with $(\phi_1(\mathbf{r}), \phi_2(\mathbf{r})) = (\Phi_1 + \varphi_1(\mathbf{r}), \Phi_2 + \varphi_2(\mathbf{r}))$ and expand $F$ to second order in $\varphi$.

Assuming that the statistical weight of thermal fluctuations is proportional to $\exp(-F)$, calculate approximately the correlation function

$$\langle\varphi_1(\mathbf{r})\varphi_1(\mathbf{0}) + \varphi_2(\mathbf{r})\varphi_2(\mathbf{0})\rangle$$

by evaluating a Gaussian functional integral. How does your result depend on the dimensionality $d$ of the system?