TOPOLOGY PROBLEMS (revised)

Please hand in your answers to Thomas Pickup in his Theoretical Physics pigeon-hole before 11am 21 March for marking.

Q1. Consider a single scalar field in 1+1 (Minkowski) dimensions with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - U(\varphi)$$

and hence energy

$$E = \int dx \left\{ \frac{1}{2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + U(\varphi) \right\}.$$

Consider the time-independent solutions of the equations of motion. Such solutions satisfy $\frac{\partial\varphi}{\partial t} = 0$ and (from Hamilton's variational principle)

$$\delta \int_{-\infty}^{+\infty} dx \left\{ \frac{1}{2}\left(\frac{\partial\varphi}{\partial x}\right)^2 + U(\varphi) \right\} = 0.$$

We can rename the variables $x \to t$, $\varphi \to x$ and then this becomes

$$\delta \int_{-\infty}^{+\infty} dt \left\{ \frac{1}{2}\left(\frac{dx}{dt}\right)^2 + U(x) \right\} = 0.$$

So the problem is equivalent to solving for the motion of a particle in a potential $V = -U$.

Hence (or otherwise) show that:

a) if

$$U = \frac{\lambda}{2}\left(\varphi^2 - a^2\right)^2$$

there is a soliton

$$\varphi = a\tanh(\mu x),$$

where $\mu^2 = a^2\lambda$, with energy

$$E = \frac{4\mu^3}{3\lambda}.$$

b) if

$$U = \frac{\alpha}{\beta^2}\left(1 - \cos\beta\varphi\right)$$

there is a soliton

$$\varphi = \frac{4}{\beta} \tan^{-1} \exp(\alpha^{1/2} x)$$

with energy

$$E = \frac{8\alpha^{1/2}}{\beta^2}.$$

Q2. Recall the Abelian-Higgs model in 1+1 (Euclidean) dimensions:

$$\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^2 + |D_\mu \varphi|^2 + \frac{\lambda}{4} \left(\varphi^* \varphi - a^2 \right)^2$$

where

$$D_\mu \varphi = (\partial_\mu + ieA_\mu)\, \varphi$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and φ is a complex scalar field. This theory has instantons which, in polar coordinates, typically behave like

$$\lim_{r \to \infty} \varphi(r, \theta) = a \exp(i\alpha(\theta))$$
$$\lim_{r \to \infty} A_\mu(r, \theta) = -\frac{1}{e} \partial_\mu \alpha(\theta)$$

where $\alpha(2\pi) = \alpha(0) + 2\pi$.

Suppose we now consider the quite different theory obtained by removing the scalar field from $\mathcal{L}_E$, i.e the theory with only the gauge field, $\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^2$.
Consider the gauge field given in the above instanton field configuration,
i.e.$\lim_{r \to \infty} A_\mu(r, \theta) = -\frac{1}{e} \partial_\mu \alpha(\theta)$.
Does it constitute a topologically non-trivial instanton of the pure gauge theory – does it belong to a non-trivial homotopy class or not?

Q3. Once again, the Abelian-Higgs model in 1+1 (Euclidean) dimensions:

$$\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^2 + |D_\mu \varphi|^2 + \frac{\lambda}{4} \left(\varphi^* \varphi - a^2 \right)^2$$

where

$$D_\mu \varphi = (\partial_\mu + ieA_\mu)\, \varphi$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and φ is a complex scalar field. Fields of finite action will satisfy

$$\lim_{r\to\infty} \varphi(r,\theta) = a\exp(i\alpha(\theta))$$

$$\lim_{r\to\infty} A_\mu(r,\theta) = -\frac{1}{e}\partial_\mu\alpha(\theta)$$

where the single-valuedness of the field φ implies that $\alpha(2\pi) = \alpha(0) + 2\pi n_W$ and n_W is the integer winding of the map from the circle at ∞ to U(1).

Show that we can write n_W as a contour integral around the circle at $r = \infty$:

$$n_W = -\frac{e}{2\pi}\oint dx_\mu A_\mu$$

and hence show that

$$n_W = -\frac{e}{4\pi}\int d^2x \epsilon_{\mu\nu}F_{\mu\nu}.$$

This tells us that if $n_W \neq 0$ then there must be some region of space-time where $F_{\mu\nu} \neq 0$. For the instanton – the minimum action field with $n_W = 1$ – this region will be its 'core'.

Note that this enables us to write the topological winding, which is a global property of the fields, in terms of a local gauge-invariant density:

$$n_W = \int d^2x n(x)$$

where

$$n(x) = -\frac{e}{4\pi}\epsilon_{\mu\nu}F_{\mu\nu}$$

If we define the topological current, G_μ, by

$$\partial_\mu G_\mu = n(x)$$

what is G_μ in terms of A_ν ?

Q4. Consider again the Abelian-Higgs model of the previous question, which contains scalar fields of charge $\pm e$ coupled to an electromagnetic potential in one spatial dimension. In the lectures we showed how one could calculate the vacuum energy, $E(\theta)$, in the dilute gas approximation.

(a) Use the dilute gas approximation to show that instantons produce a linearly confining potential between heavy sources of charge $\pm q$, i.e. $V(r) \sim \sigma(q)r$ when the distance between the charges, r, becomes large. Calculate the string tension σ.

Hints:

Consider the integral around a rectangular $\mathcal{C} \equiv R \times T$ contour of the exponential of the gauge potential:

$$W(R,T) \equiv e^{iq \int_C A_\mu dx_\mu}$$

This is called a Wilson loop. One can show (you can assume) that for large enough T

$$\langle e^{iq \int_C A_\mu dx_\mu} \rangle \propto e^{-V(R)T}$$

where $V(R)$ is the potential between static sources of charge $\pm q$ a distance R apart. So if for large values of R we find that

$$\langle e^{iq \int_C A_\mu dx_\mu} \rangle \propto e^{-\sigma RT}$$

then this means that the potential grows linearly, $V(R) \sim \sigma R$, and in that case we have shown linear confinement. So calculate $\langle e^{iq \int_C A_\mu dx_\mu} \rangle$ in the dilute gas approximation and show that

$$\langle e^{iq \int_C A_\mu dx_\mu} \rangle = e^{-2KRTe^{-S_I}(1-\cos 2\pi \frac{q}{e})}$$

so that the confining string tension is

$$\sigma = 2Ke^{-S_I}(1 - \cos 2\pi \frac{q}{e}).$$

To calculate the Wilson loop expectation value, recall that for any quantity $\mathcal{O}$

$$\langle \mathcal{O} \rangle = \frac{\int D\varphi \mathcal{O} e^{-S_E}}{\int D\varphi e^{-S_E}}$$

where S_E is the Euclidean action. In our case $\mathcal{O}$ is the Wilson loop and you should calculate the numerator and denominator in the dilute gas approximation where the instantons are the only fluctuations of the fields. To calculate the numerator, consider separately the effects of the instantons inside and outside the Wilson loop. Assume that the probability that an instanton core overlaps with the perimeter of the Wilson loop is negligible, so that an instanton is always entirely inside or entirely outside the perimeter of the loop. You will find it convenient to use an expression for n_W that occurs in the previous question.

(b) The expression for $\sigma(q)$ is zero for $q = ne$. Can you think of a simple dynamical reason for this?

(c) Suppose we add to the action the topological term

$$\delta \mathcal{S}_E = i\theta \frac{e}{4\pi} \int d^2x \epsilon_{\mu\nu} F_{\mu\nu} = i\theta Q$$

where Q is the total topological charge of the field: $Q \equiv n_W = n_I - n_{\bar{I}}$, where $n_I, n_{\bar{I}}$ are the number of instantons, anti-instantons in the field. Does this alter the expression for σ and, if so, how? Any simple interpretation of what is going on?

Q5. Recall the proof in the lectures of Derrick's Theorem for a theory with scalar fields in d_s spatial dimensions. We start by supposing that $\phi_s(\vec{x})$ is the time-independent soliton solution, in which case it will minimise the energy:

$$\frac{\delta E[\phi]}{\delta \phi} = 0 \qquad \text{for } \phi = \phi_s.$$

We then define a family of fields labelled by a parameter λ by:

$$\phi(\vec{x}, \lambda) = \phi_s(\lambda \vec{x}).$$

so that $\phi(\vec{x}, \lambda = 1) = \phi_s(\vec{x})$. We split the energy into $E = V_1 + V_2$ where $V_1 \geq 0$ contains the integral over space of the spatial derivatives, and $V_2 \geq 0$ contains the integral over space of the potential. By a simple change of variables we then see that

$$E(\lambda) = \lambda^{2-d_s} V_1(\lambda = 1) + \lambda^{-d_s} V_2(\lambda = 1)$$

It is thus clear that for $d_s > 2$ we can decrease $E(\lambda)$ by increasing λ – but this contradicts the assumption that $E(\lambda = 1)$ is a minimum. The only way out is if $V_1(\lambda = 1) = V_2(\lambda = 1) = 0$ which means that $\phi_s(\vec{x})$ is a constant field taking a value in the minimum of the potential. This field has zero energy density everywhere, and so is not a non-dissipative solution i.e. not a soliton with our definition.

For $d_s = 2$ there is no λ factor in front of V_1 and one needs only $V_2(\lambda = 1) = 0$, i.e. that the field $\phi_s(\vec{x})$ takes values in the minimum of U for all $\vec{x}$, but need not be constant.

Three questions.

a) Consider the Abelian-Higgs model in 2 space dimensions. We claimed that this theory does have a time-independent soliton, with a non-trivial core in which $U(\phi_s(\vec{x}))$ is far from its minimum. How does it evade Derrick's theorem? (Note that the total energy in this theory may be written as

$$E = \int d^{d_s}x \{\frac{1}{2}(\partial_0 A_i)^2 + \partial_0 \phi^\dagger \partial_0 \phi\} + \int d^{d_s}x \{\frac{1}{4} F_{ij}^2 + D_i \phi^\dagger D_i \phi + U(\phi)\}$$

in the $A_0 = 0$ gauge. Here $i, j = 1, ..., d_s$ are spatial indices.)

b) Consider the $O(3)$ spin model in 2+1 dimensions. This model has three real scalar fields, $\{\phi_i; i = 1, .., 3\}$, that are constrained to unit length

$$\phi_1^2 + \phi_2^2 + \phi_3^2 = 1$$

and that have a Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 + \frac{1}{2}\partial_\mu\varphi_3\partial^\mu\varphi_3.$$

If we identify the circle at spatial infinity then the spatial plane becomes a sphere and the mapping from this sphere to $O(3)$ has a winding number and so there exists the possibility of topological solitons. What happens to these (candidate) solitons if we apply the scaling argument used in the proof of Derrick's theorem? What happens if we add an arbitrarily small amount of energy to such a soliton – is it non-dissipative? What does this suggest about the existence of solitons in the quantised theory?

c) Suppose we are in 2+1 dimensions and we have 3 scalar fields with the familiar potential

$$U = \lambda\left(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 - a^2\right)^2.$$

If we increase λ while keeping a fixed, it is clear that the fields fluctuate less and less about $\varphi_1^2 + \varphi_2^2 + \varphi_3^2 = a^2$ and so we appear to be approaching the $O(3)$ spin model of Q5(b) except for the trivial difference that the spin has length a rather than 1. For a fixed value of λ, very large, does the classical theory contain solitons, i.e. non-dissipative solutions of the equations of motion, and in particular of the kind discussed in Q5(b)? If so, what do you expect will be the fate of any such soliton in the quantised theory? What do you expect is going to be the effect of decreasing λ?

Q6. Are there instantons in SU(2) gauge theory for the number of Euclidean space-time dimensions $\neq 4$? Why or why not? (Hint: Use a scaling argument as for Derrick's theorem, but allow yourself to scale the field as well.)