Lecture notes: 3rd year fluids

Section B: Inviscid (or Ideal) Flow

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B.1 Kelvin's circulation theorem

The definition of circulation is

$$\Gamma = \oint_{C(t)} \mathbf{u} \cdot d\boldsymbol{\ell}$$

where $C(t)$ is a closed circuit following the flow.

Kelvin's circulation theorem states

$$\frac{D\Gamma}{Dt} = 0$$

for an *inviscid, incompressible* fluid (and any forces have to be conservative).

Proof

$$\frac{D}{Dt} \oint_{C(t)} \mathbf{u} \cdot d\boldsymbol{\ell} = \oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\boldsymbol{\ell} + \oint_{C(t)} \mathbf{u} \cdot \frac{D(d\boldsymbol{\ell})}{Dt}. \tag{1}$$

Considering the first term on the rhs of Eq. (1), and using the Euler equation,

$$\oint_{C(t)} \frac{D\mathbf{u}}{Dt} \cdot d\boldsymbol{\ell} = -\frac{1}{\rho} \oint_{C(t)} \nabla p \cdot d\boldsymbol{\ell} = -\frac{1}{\rho} \oint_{C(t)} dp = 0$$

because $\nabla p \cdot d\boldsymbol{\ell}$ is an exact derivative.

The second term on the rhs can be written

$$\oint_{C(t)} \mathbf{u} \cdot \frac{D(d\boldsymbol{\ell})}{Dt} = \oint_{C(t)} \mathbf{u} \cdot d\mathbf{u} = \frac{1}{2} \oint_{C(t)} d(\mathbf{u}^2) = 0$$

because $d(\mathbf{u}^2)$ is an exact derivative (see Figure 1).

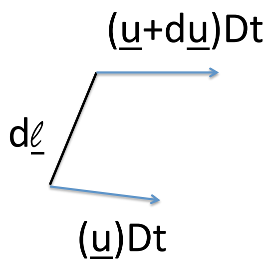

Figure 1: Recall that $d\boldsymbol{\ell}$ is a line element. In a time Dt one end moves by $\mathbf{u}\ Dt$ and the other end by $(\mathbf{u} + d\mathbf{u})\ Dt$. So $d\boldsymbol{\ell}$ changes by $d\mathbf{u}\ Dt$ and $\frac{D(d\boldsymbol{\ell})}{Dt} = d\mathbf{u}$.

Using Stokes theorem it is apparent that the circulation is related to the vorticity:

$$\Gamma = \oint_{C(t)} \mathbf{u} \cdot d\boldsymbol{\ell} = \int_S \mathrm{curl}\, \mathbf{u} \cdot \hat{\mathbf{n}}\ \ dS = \int_S \boldsymbol{\omega} \cdot \hat{\mathbf{n}}\ \ dS.$$

where S is a surface spanning C.

Consequences of Kelvin's circulation theorem:

- If $\Gamma = 0$ round any closed curve, it remains zero => an irrotational fluid remains irrotational (if it is inviscid and incompressible).
- a flow field can often usefully be modelled as a collection of vortex tubes with non-zero circulation, with regions of irrotational flow separating them. Examples are smoke rings, a tornado.
- Γ is the same for all cross sections of a vortex tube => vortex stretching => incease in ω.
- vortex lines must form closed loops or terminate on a boundary.

B2 The dynamics of vortex tubes

Kelvin's theorem implies that vortex tubes and rings are persistent structures.

a. How do they evolve in time?

The vorticity equation is (see Section A9(c))

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

If the fluid is inviscid ($\nu = 0$) and initially irrotational ($\boldsymbol{\omega} = 0$) then $\frac{D\boldsymbol{\omega}}{Dt} = 0$ and the vorticity remains zero in agreement with Kelvin's circulation theorem.

What happens if the fluid is inviscid ($\nu = 0$) and initially $\boldsymbol{\omega} \neq 0$? To understand how $\boldsymbol{\omega}$ evolves, we consider a vortex tube along $\hat{\mathbf{z}}$ so that $\boldsymbol{\omega} = (0, 0, \omega_z)$. Writing the total derivative of the vorticity is terms of a componemt along $\boldsymbol{\omega}$ and a component perpendicular to $\boldsymbol{\omega}$

$$\frac{D\boldsymbol{\omega}}{Dt} = \omega_z \frac{\partial u_z}{\partial z} \hat{\mathbf{z}} + \omega_z \frac{\partial u_\perp}{\partial z} \hat{\mathbf{e}}_\perp$$

The first term on the rhs describes vortex stretching. If $\frac{\partial u_z}{\partial z} > 0$ the vortex tube elongates along $\hat{\mathbf{z}}$. This means that it becomes thinner and, to preserve the circulation, the vorticity increases.

The second term on the rhs describes vortex twisting. Vorticity in the perpendicular direction is created from vorticity originally along $\hat{\mathbf{z}}$ by gradients in $u_\perp$.

b. How is vorticity created?

• velocity gradients at walls due to the no-slip boundary conditions result in viscous boundary layers which can become unstable to vortex formation.

• non-conservative forces. eg the Coriolis force which is a consequence of the rotation of the earth and responsible for the circulation of the atmosphere.

c. How is vorticity destroyed?

• Vorticity diffuses away as a result of viscosity.

B3 Irrotational Flow

If the flow is *irrotational*, $\text{curl}\,\mathbf{u} = 0$ and we can define a velocity potential ϕ by

$$\mathbf{u} = \text{grad}\,\phi.$$

If the flow is also *incompressible*, $\text{div}\,\mathbf{u} = 0$ and

$$\text{div}\,\mathbf{u} = \nabla^2\phi = 0.$$

This is true for all irrotational flows, but it is most useful close to the inviscid limit where the flow remains irrotational.

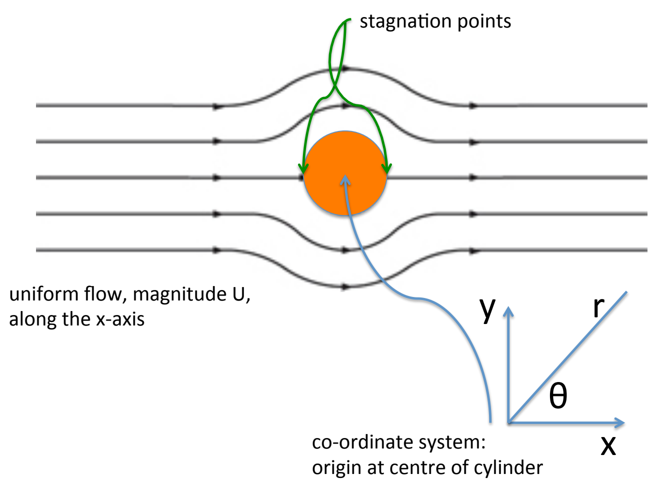

Figure 2: Irrotational flow around a cylinder.

As an example we use techniques familiar from electrostatics to find the flow around a cylinder, radius a. The boundary conditions are:

• as $r \to \infty$, $\phi \to Ux = Ur\cos\theta$ corresponding to uniform flow of magnitude U along $\hat{x}$ at infinity.

• $u_r = 0$ on $r = a$, the no penetration condition.

(NB there is no boundary condition on u_θ in this approximation – we are working in the inviscid limit which cannot get no-slip correct.)

To match the boundary conditions use the '$\cos\theta$' solutions of the Laplace equation in cylindrical co-ordinates

$$\phi = C_1 r \cos\theta + \frac{C_2}{r}\cos\theta.$$

Putting in the boundary conditions gives

$$\boxed{\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta.}$$

To find the streamlines we can follow the path
velocity potential $\phi \to$ velocity $\mathbf{u} \to$ streamfunction $\psi \to$ streamlines, ψ =constant.

The velocity field is

$$\begin{aligned} u_r &= \frac{\partial\phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta \equiv \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \\ u_\theta &= \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta = -\frac{\partial\psi}{\partial r}. \end{aligned} \tag{2}$$

Therefore, by inspection, the streamlines are

$$\psi = U\left(r - \frac{a^2}{r}\right)\sin\theta = Uy\left(1 - \frac{a^2}{r^2}\right) = \text{constant}.$$

There are *stagnation points* where the velocity is zero on the surface of the cylinder at $\theta = 0, \pi$.

In the next section we will use Bernoulli's equation to calculate the pressure distribution around the cylinder.

B4 The Bernoulli equation

The aim of this section is to show that the Bernoulli function

$$H = \frac{p}{\rho} + \frac{\mathbf{u}^2}{2} + \chi$$

is constant along a streamline. χ is the potential associated with any conservative force that is acting on the fluid.

- We will need the vector identity

$$(\mathbf{u}\cdot\nabla)\,\mathbf{u} = (\nabla\wedge\mathbf{u})\wedge\mathbf{u} + \nabla\left(\frac{\mathbf{u}^2}{2}\right). \tag{3}$$

• We shall require that $\mathbf{g}$ is a conservative force so that we can write it in terms of a potential $g\hat{\mathbf{z}} = \nabla\chi$ where $\chi = gz$.

The proof starts from the Euler equation (ie we are assuming an inviscid fluid) and we assume constant density

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p - \mathbf{g} = -\nabla\left(\frac{p}{\rho} + \chi\right)$$

Using equation (3)

$$\frac{\partial \mathbf{u}}{\partial t} + (\nabla\wedge\mathbf{u})\wedge\mathbf{u} = -\nabla\left(\frac{p}{\rho} + \frac{\mathbf{u}^2}{2} + \chi\right) \equiv -\nabla H.$$

If the flow is steady

$$(\nabla\wedge\mathbf{u})\wedge\mathbf{u} = -\nabla H. \qquad (4)$$

Taking the dot product with $\mathbf{u}$:

$$0 = -(\mathbf{u}\cdot\nabla)H$$

so H is constant along a streamline (but can vary between streamlines).

If the flow is also irrotational $\nabla\wedge\mathbf{u} = 0$. So, from equation (4)

$$\nabla H = 0$$

so H is constant everywhere.

(a) Venturi gauge

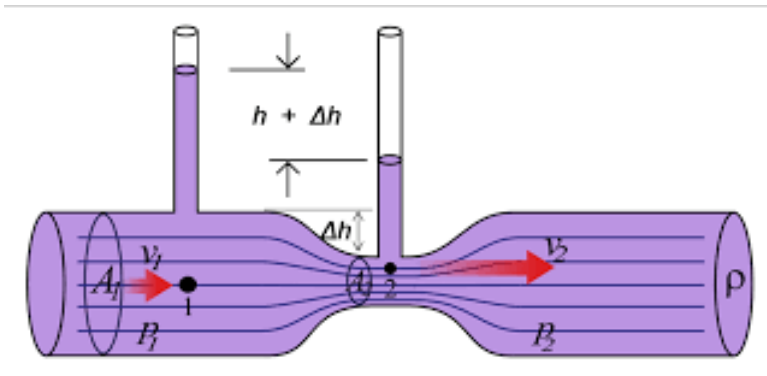

Figure 3: H is constant along streamlines. In the narrower part of the tube the velocity increases, so the pressure decreases.

(b) Considering the cylinder in Figure 2, the flow is irrotational so H is constant throughout the flow. Therefore on the surface of the cylinder

$$\frac{\mathbf{u}^2(a,\theta)}{2} + \frac{p(a,\theta)}{\rho} = \text{constant}.$$

From equation (2)

$$u_r(a,\theta) = 0 \quad \text{as expected}, \quad u_\theta(a,\theta) = -2U\sin\theta.$$

so

$$\frac{4U^2\sin^2\theta}{2} + \frac{p(a,\theta)}{\rho} = \text{constant} = \frac{p(a,0)}{\rho}.$$

where the rhs is a sensible choice of a reference pressure. Rearranging gives

$$p(a,\theta) = p(a,0) - 2\rho U^2 \sin^2\theta.$$

The pressure is lower on the top and bottom of the cylinder than at the sides because the fluid is moving faster.
Note, however, that
the pressure is the same top and bottom → no net *lift*.
the pressure is the same right and left → no net *drag*.

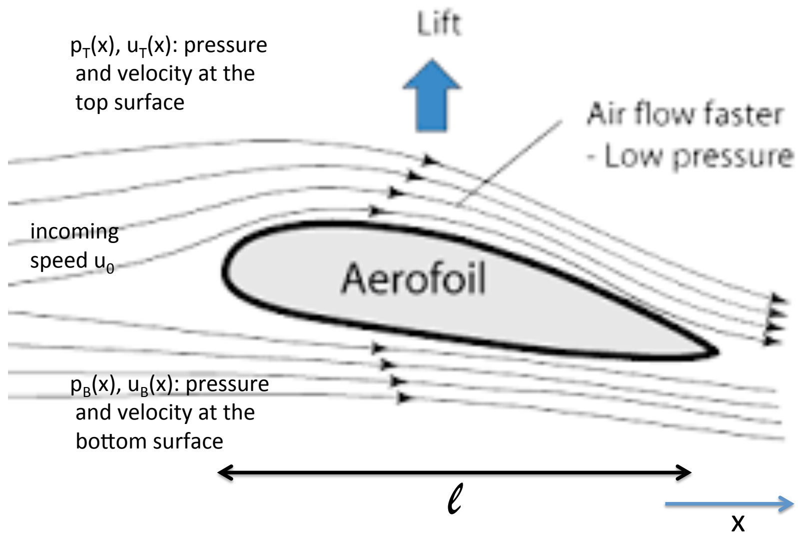

Figure 4: Flow over an aerofoil.

B5 Lift and the Kutta-Joukovski theorem

To get lift a non-zero circulation is needed.

Consider a thin, 2D aerofoil inclined at a small angle to the flow direction (Figure 4).

For irrotational flow Bernoulli's theorem gives

$$\frac{p_B}{\rho} + \frac{u_B^2}{2} = \frac{p_T}{\rho} + \frac{u_T^2}{2}.$$

where symbols are defined in Figure 4. Rearranging

$$p_B - p_T = \frac{\rho}{2}(u_T^2 - u_B^2) = \frac{\rho}{2}(u_T + u_B)(u_T - u_B) \approx \rho u_0(u_T - u_B).$$

The lift per unit span is

$$L = \int_0^\ell (p_B - p_T)dx = \rho u_0 \int_0^\ell (u_T - u_B)dx.$$

The circulation around the boundary of the aerofoil is

$$\Gamma = \int_0^\ell (u_B - u_T)dx$$

which leads to the Kutta-Joukovski theorem in 2D

$$L = -\rho u_0 \Gamma.$$

But why can there be circulation in an irrotational fluid? This is OK because Γ is the circulation around any loop containing the aerofoil; if the loop does not enclose the aerofoil, $\Gamma = 0$.

• When an aerofoil starts to move a *starting vortex* is formed near the trailing edge because of viscous effects in the boundary layer. The starting vortex is left behind, leaving the aerofoil with a net circulation.

• In 3D it is not possible to just have a starting vortex as vortex tubes must start and end on boundaries. The vortex structure around a plane is:

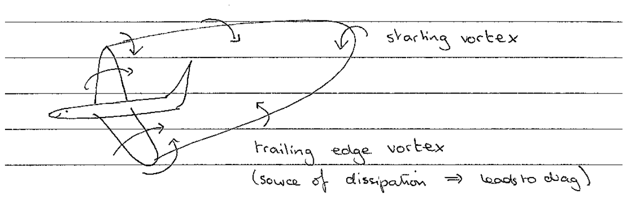

Figure 5: Vortices around an aircraft.

• If the angle of the aerofoil with respect to the direction of motion is too large the streamlines no longer follow the boundary of the aerofoil. This leads to boundary layer separation and turbulent flow above the wing and a consequent decrease in lift. This is stalling: when a plane lands or takes off wing flaps are used to prevent it.