Lecture notes: 3rd year fluids

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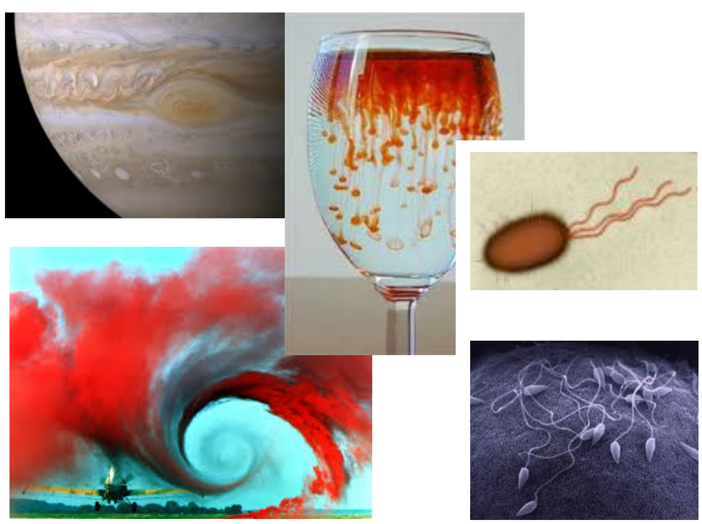
Preamble

We will be discussing the Navier Stokes equation

$$\rho\{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\} = -\nabla p + \eta \nabla^2 \mathbf{u}. \quad (1)$$

It is amazing that such a seemingly simple equation can be used to describe



how fluids move across an enormous range of length scales.

A The Navier-Stokes Equation

A.1 Vectors: reminders and identities

$$\begin{aligned}
\operatorname{grad}\phi \equiv \nabla\phi &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \\
\operatorname{div}\mathbf{u} \equiv \nabla\cdot\mathbf{u} &= \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \\
\operatorname{curl}\mathbf{u} \equiv \nabla\wedge\mathbf{u} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} \\
\nabla^2\phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}
\end{aligned}$$

$\nabla^2\mathbf{u}$ is a **vector** with components $\left(\nabla^2 u_x, \nabla^2 u_y, \nabla^2 u_z\right)$

$(\mathbf{u}\cdot\nabla)\mathbf{u}$ is a **vector** with x-component

$$\left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)u_x, \quad \text{etc.}$$

N.B.1 these formulas are different in different co-ordinate systems — see e.g. Acheson appendix.

N.B.2 there are lots of useful vector identities

e.g. $\nabla(\mathbf{F}\cdot\mathbf{G}) = \mathbf{F}\wedge(\nabla\wedge\mathbf{G}) + \mathbf{G}\wedge(\nabla\wedge\mathbf{F}) + (\mathbf{F}\cdot\nabla)\mathbf{G} + (\mathbf{G}\cdot\nabla)\mathbf{F}$

— also listed in Acheson appendix

Divergence theorem

$$\int_S \mathbf{u}\cdot\hat{\mathbf{n}}\, dS = \int_V \nabla\cdot\mathbf{u}\, dV \tag{2}$$

where S is the closed surface surrounding a volume V.

An equivalent statement of the divergence theorem is

$$\int_S \phi\hat{\mathbf{n}}\, dS = \int_V \nabla\phi\, dV. \tag{3}$$

Stokes theorem

$$\oint_C \mathbf{u} \cdot \mathbf{ds} = \int_s (\nabla \wedge \mathbf{u}) \cdot \hat{\mathbf{n}} \, dS$$

where S is an open surface spanning a closed curve C.

A.2 Continuity equation

The continuity equation is a statement of conservation of mass.

Consider a volume V. Conservation of mass implies:

$$\begin{aligned} \text{decrease of mass in } V &= \text{total mass flux out of } V \\ -\frac{\partial}{\partial t} \int \rho \, dV &= \int_S \rho \mathbf{u} \cdot \hat{\mathbf{n}} \, dS \\ -\frac{\partial}{\partial t} \int \rho \, dV &= \int_V \nabla \cdot (\rho \mathbf{u}) \, dV \end{aligned}$$

This is true for all $V \quad \Rightarrow$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

If the fluid is $\boxed{\text{incompressible, } \rho \text{ is constant}}$ so

$$\boxed{\nabla \cdot \mathbf{u} = 0.}$$

(Incompressible implies that pressure variations in the flow do not significantly alter the density. Liquids have a small compressibility so this is usually a very a good approximation. For gases it is not so obvious, but often the pressure variations are sufficiently small that it remains a good approximation. The quantitative criterion is $\mathrm{Ma} = u/c_\mathrm{s} \ll 1$, where Ma is the Mach number, u is the flow speed, and c_s is the speed of sound. See eg Tritton 5.8.)

A.3 Material derivative

Let f be a quantity associated with a 'fluid particle'. How does it change with time?

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{d}{dt} f\left(x(t), y(t), z(t), t\right).$$

Therefore, using the chain rule,

$$\begin{aligned}\frac{\mathrm{D}f}{\mathrm{D}t} &= \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t} \\ &= u_x\frac{\partial f}{\partial x} + u_y\frac{\partial f}{\partial y} + u_z\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t} \\ &= (\mathbf{u}\cdot\nabla)f + \frac{\partial f}{\partial t}.\end{aligned}$$

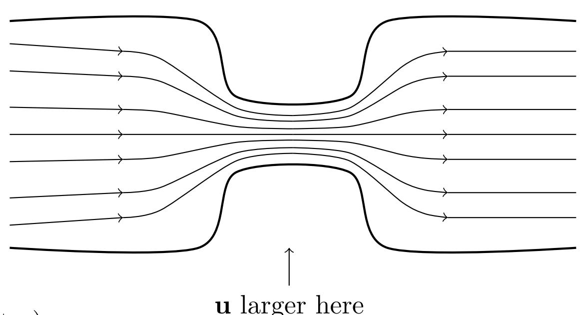
$\frac{\mathrm{D}f}{\mathrm{D}t}$ is called the *material derivative*, the rate of change of f following the fluid.

Physically the material derivative is the rate of change of f that an observer moving with the fluid would measure at any particular location in space and instant in time where the derivative is evaluated.

So the acceleration of a fluid particle is

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} = \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}.$$

An example of the acceleration of a particle in a steady ($\frac{\partial \mathbf{u}}{\partial t} = 0$) flow:



(constant ρ)

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} \neq 0 \text{ even though } \frac{\partial \mathbf{u}}{\partial t} = 0.$$

A.4 Euler equation

The Euler equation is a statement of conservation of momentum (ie Navier-Stokes with zero viscosity).

Newton's law for a fluid element (assuming an incompressible fluid so ρ is constant) is

$$\begin{aligned}
\text{“mass} \times \text{acceleration”} &= \int_V \rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t}\, dV \\
\text{“force”} &= -\int_S p\hat{\mathbf{n}}\, dS + \int_V \mathbf{f}\, dV \\
&= -\int_V \nabla p\, dV + \int_V \mathbf{f}\, dV
\end{aligned}$$

where we have used the divergence theorem, eq. (3), p is the pressure and $\mathbf{f}$ is a force per unit volume, sometimes called a body force.

True for all V so

$$\begin{aligned}
\rho \frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} &= -\nabla p + \mathbf{f} \\
\text{or, equivalently,} \quad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\frac{1}{\rho}\nabla p +, \frac{\mathbf{f}}{\rho}.
\end{aligned}$$

NB if the force is gravity

$$\frac{\mathbf{f}}{\rho} = -g\mathbf{k}$$

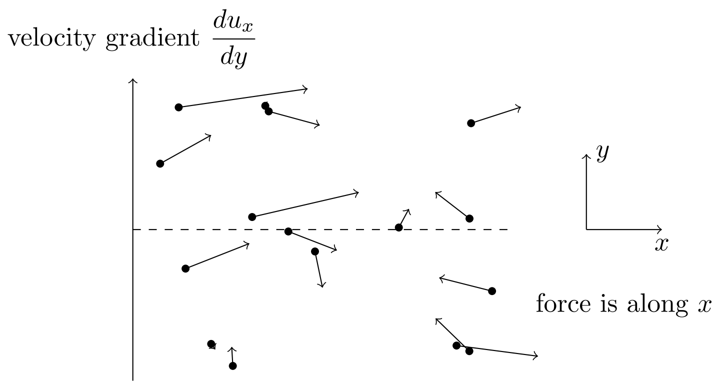
(assuming $\mathbf{k}$ is the upwards unit vector).

A.5 Viscosity and the Navier-Stokes equation

... but there are other forces acting due to velocity gradients that we have ignored so far. Velocity gradients lead to momentum transfer, ie to forces.

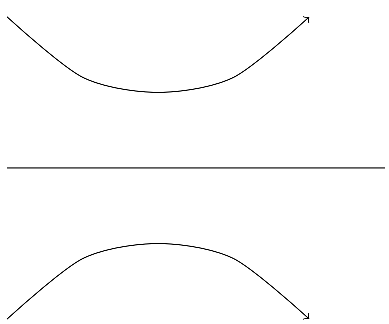
(a) the physics (cf kinetic theory)

(i) kinetic contribution to viscosity: dominant in a gas



molecules crossing plane from above carry more $x-$momentum than those crossing from below

(ii) momentum transfer due to intermolecular forces



collisional contribution to viscosity: dominant in a liquid

(b) the maths (outline only)

So far we have the Euler equation (ignoring external forces for now and writing in component form)

$$\frac{\partial u_i}{\partial t} + (\mathbf{u} \cdot \nabla) u_i = -\frac{1}{\rho} \frac{\partial p}{\partial x_i} = \frac{1}{\rho} \frac{\partial}{\partial x_j} \{-p\delta_{ij}\}.$$

We generalise this by writing

$$\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla)u_i = \frac{1}{\rho}\frac{\partial}{\partial x_j}\{\sigma_{ij}\} \tag{4}$$

$$\text{where}\quad \sigma_{ij} = -p\delta_{ij} + \eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{5}$$

σ_{ij} is the *stress tensor*

η is the *dynamic viscosity*

σ_{ij} is the ith component of the stress (force per unit area) on an element of surface with normal in direction j.

So the ith component of stress on an element of surface area with normal $\hat{\mathbf{n}}$ is

$$t_i = \sigma_{ix}n_x + \sigma_{iy}n_y + \sigma_{iz}n_z = \sigma_{ij}n_j$$

$$\boxed{\mathbf{t} = \boldsymbol{\sigma}\mathbf{n}}$$

Why does the viscous term take the form $\eta\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$?

- must depend on velocity gradients
- we assume that gradients are small so the dominant contribution is linear in first derivatives of the velocity
- this simple form assumes incompressibility
- the allowed combinations of gradients is restricted by symmetry (e.g. interchanging i and j cannot change the physics)

Putting σ_{ij} into the Navier-Stokes equation (4)

$$\frac{\partial u_i}{\partial t} + (\mathbf{u}\cdot\nabla)u_i = \frac{1}{\rho}\left(-\frac{\partial p}{\partial x_i} + \eta\frac{\partial}{\partial x_j}\left\{\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right\}\right).$$

But note that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right) = \nabla^2 u_i,$$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right) = \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) = 0 \quad \text{for an incompressible fluid.}$$

So, returning to vector notation,

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \frac{\eta}{\rho}\nabla^2 \mathbf{u}.$$

$\nu = \eta/\rho$ is the *kinematic viscosity.*

A.6 Comments on the validity of the Navier-Stokes equation:

1. Navier-Stokes is a *continuum* equation, ie it is written in terms of continuous variables or fields ρ, $\mathbf{u}$. We assume that we can define the variables at a point in space at a given time $\rho(\mathbf{r}, t)$; $\mathbf{u}(\mathbf{r}, r)$ but really we are associating them with a region in space of size l, say. If l is too small the number of molecules in the region and their average velocity will fluctuate widely so the continuum limit (or, equivalently, the hydrodynamic limit) will not work. N-S appears to work well down to surprisingly small length scales for a fluid ($\sim 50\,\text{nm}$ & below). For rarefied gases need $l \gg \lambda$, where λ is the mean free path, and there are cases when N-S fails.

2. $$\sigma_{ij} = -p\delta_{ij} + \eta \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

   is a *constitutive relation* between velocity gradients and the stress tensor. It can be taken as the definition of a Newtonian fluid.

3. When does N-S fail.?

   - viscoelastic fluids such as liquid crystals or polymer solutions have a memory $\Rightarrow$ need an extra time scale.
   - steep velocity gradients mean that higher order derivatives will be important.
   - fluids that are magnetic or have free charges are described by more complicated equations.
   - if the flow of heat is important an extra temperature field is needed.
   - relativistic effects are ignored.

4. Coefficients like viscosity η are numbers that need to be measured. Calculating them requires a microscopic theory (like kinetic theory).

A.7 Visualising the flow field

velocity field:
plot of vectors $\mathbf{u}(\mathbf{r})$ at time t
gives information about direction and magnitude of the flow

streamline:
curves tangent to $\mathbf{u}$ at time t

In *steady flow* ($\frac{\partial \mathbf{u}}{\partial t} = 0$) the streamlines do not change with time and particle paths = streamlines.

Streamlines are given by

$$\frac{dx}{u_x} = \frac{dy}{u_y} = \frac{dz}{u_z}$$

because (dx, dy, dz) is locally $\propto$ $\mathbf{u}$.

Streamlines cannot intersect.

Each streamline gives information about local direction of the flow; and the density of streamlines gives information about magnitude of the flow.

N.B.1 the rate of change of f of a fluid particle following the fluid is

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \frac{\partial f}{\partial t} + \mathbf{u} \cdot \nabla f.$$

In steady flow $\frac{\partial f}{\partial t} = 0$ and

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \mathbf{u} \cdot \nabla f.$$

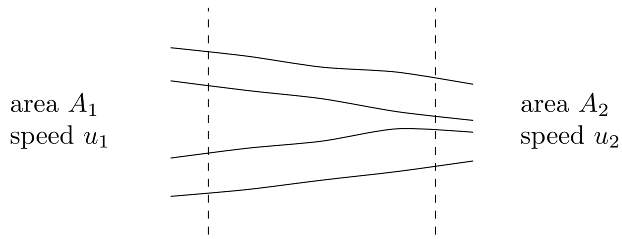
In steady flow particles move along streamlines so this is the rate of change of f along a streamline, and if

$$\frac{\mathrm{D}f}{\mathrm{D}t} = \mathbf{u} \cdot \nabla f = 0$$

f is constant along a streamline (but can vary from streamline to streamline).

N.B.2 streamlines can be measured experimentally by using tracer particles which have to be small enough not to disturb the flow but large enough that Brownian motion is insignificant $\sim 0.5\,\mu\mathrm{m}$.

streamtubes:
bundles of streamlines



rate at which mass entering at 1 = $\rho_1 A_1 u_1$
rate at which mass leaving at 2 = $\rho_2 A_2 u_2$

so if the flow is incompressible

$$Au = \text{constant}$$

streamtube becomes smaller $\Leftrightarrow$ flow accelerates

stream function ψ

Can be defined if flow is *incompressible* and *two-dimensional* by writing

$$u_x = \frac{\partial \psi}{\partial y}, \quad u_y = -\frac{\partial \psi}{\partial x}$$

so that the incompressibility condition $\nabla \cdot \mathbf{u} = 0$ is automatically satisfied.

Why is ψ useful? Consider

$$(\mathbf{u} \cdot \nabla)\psi = u_x \frac{\partial \psi}{\partial x} + u_y \frac{\partial \psi}{\partial y} = 0.$$

ψ is constant along a streamline so finding ψ is equivalent to finding the streamlines.

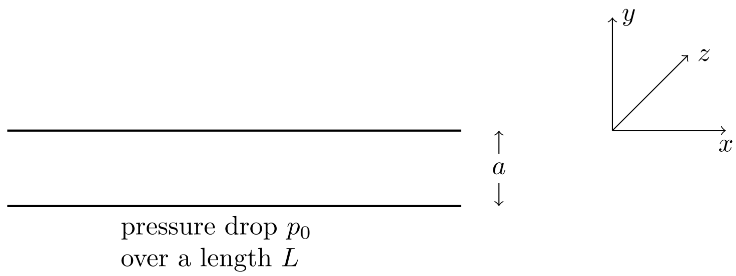
A.8 Solving N-S in a simple geometry

First we need boundary conditions. At a solid boundary $\mathbf{u} = 0$.

$u_{\text{normal}} = 0$: obvious for no source or sink of fluid at the wall.
$u_{\text{tangential}} = 0$: not obvious but an experimental fact.
This is called a no-slip boundary condition.

We shall consider pressure-driven flow down a pipe (aka channel flow, Poiseuille flow).

(i) geometry:



assume flow between 2 plates; translationally invariant in $z \Rightarrow$ makes the problem 2D

(ii) the equation:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \frac{\eta}{\rho}\nabla^2 \mathbf{u} \tag{6}$$

(iii) simplifying the equation:

- steady flow

$$\frac{\partial \mathbf{u}}{\partial t} = 0$$

- assume (motivated by symmetry arguments)

  $\partial_z \equiv 0$

  a flow field $u_x(y)$, $u_y = 0$, $u_z = 0$ so that $(\mathbf{u} \cdot \nabla)\mathbf{u} \equiv u_x \frac{\partial}{\partial x} u_x = 0$

  so (6) simplifies to $\nabla p = \eta \nabla^2 \mathbf{u}$.

  Only the x-component survives

$$\frac{\partial p}{\partial x} = \frac{p_0}{L} = \eta \frac{d^2 u_x}{dy^2}.$$

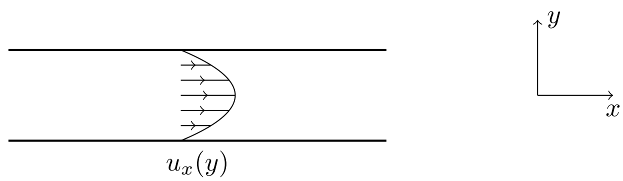
(iv) integrate and put in the boundary conditions:

$$u_x = \frac{p_0}{2\eta L}(y^2 + C_1 y + C_2).$$

$u_x = 0$ at $y = 0$ and $y = a$ so

$$u_x = \frac{p_0}{2\eta L} y(y - a)$$

giving a parabolic flow profile.



NB1 In a circular pipe of radius a

$$u_x = \frac{p_0}{4L\eta}(a^2 - r^2).$$

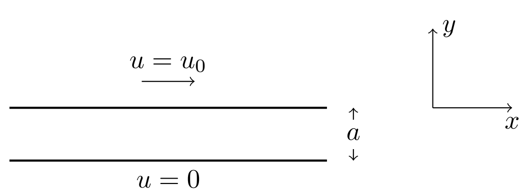
(Check using N-S in cylindrical polars.)

NB2 This solution is *laminar flow* (straight stream-lines).

For higher velocities it becomes unstable $\Rightarrow$ turbulence.

For higher velocities end effects become important too; it takes an appreciable length of pipe for flow to settle to its parabolic profile.

NB3 another simple laminar flow is *shear flow* or *Couette flow.*



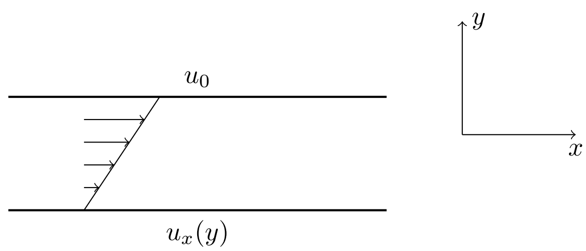
- same geometry as for Poiseuille flow

- no pressure gradient; forcing is from boundary

We need to solve

$$\eta \frac{d^2 u_x}{dy^2} = 0 \qquad \text{with} \quad u_x = u_0 \quad \text{at} \quad y = a, \qquad u_x = 0 \quad \text{at} \quad y = 0.$$

$\therefore u_x = \dfrac{u_0 y}{a}$, a linear flow profile.



A.9 The Reynolds number

(a) Non-dimensionalising Navier-Stokes

The N-S equation is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{u}. \tag{7}$$

The terms on the lhs are the *intertial* terms, the first term on the rhs is the forcing term, and the second term on the rhs is the *viscous* term.

We define dimensionless variables

$$\tilde{\mathbf{u}} = \frac{\mathbf{u}}{U}, \qquad \tilde{\mathbf{x}} = \frac{\mathbf{x}}{L}$$

where U is a velocity scale and L is a length scale. It follows that

$$\tilde{t} = \frac{Ut}{L}, \qquad \tilde{\nabla} = L\nabla, \qquad \frac{\tilde{p}}{\tilde{\rho}} = \frac{p}{U^2 \rho}.$$

Writing N-S in terms of dimensionless variables

$$\frac{U^2}{L}\frac{\partial \tilde{\mathbf{u}}}{\partial t} + \frac{U^2}{L}(\tilde{\mathbf{u}} \cdot \tilde{\nabla})\tilde{\mathbf{u}} = -\frac{U^2}{L}\frac{\tilde{\nabla}\tilde{p}}{\tilde{\rho}} + \frac{U\nu}{L^2}\tilde{\nabla}^2 \tilde{\mathbf{u}}.$$

Dividing through by U^2/L

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \tilde{\nabla})\tilde{\mathbf{u}} = -\frac{\tilde{\nabla}\tilde{p}}{\tilde{\rho}} + \frac{\nu}{LU}\tilde{\nabla}^2\tilde{\mathbf{u}} \equiv -\frac{\tilde{\nabla}\tilde{p}}{\tilde{\rho}} + \frac{1}{Re}\tilde{\nabla}^2\tilde{\mathbf{u}}$$

where

$$\boxed{Re = \frac{LU}{\nu}}$$

is the *Reynolds number*, a *dimensionless number* that characterises the flow.

For $Re \gg 1$, large length scales, high velocities, low viscosity, the inertial term dominates.
For $Re \ll 1$, small length scales, low velocities, high viscosity, the viscous term dominates.

(b) estimating the Reynolds number

stirring tea	$L \sim 10^{-2}$m	
	$U \sim 10^{-1}$ms^{-1}	'everyday' length scales for water
	$\nu \sim 10^{-6}$ m^2 s^{-1}	or air correspond to quite high Re
	Re $\sim 10^3$	
Niagara Falls	$L \sim 10$m	
	$U \sim 10$ms^{-1}	turbulent flow
	$\nu \sim 10^{-6}$ m^2 s^{-1}	
	Re $\sim 10^8$	
colloid	$L \sim 10^{-6}$m	
	$U \sim 10^{-6}$ms^{-1}	can forget about the inertial term
	$\nu \sim 10^{-6}$ m^2 s^{-1}	
	Re $\sim 10^{-6}$	

(c) examples of increasing the Reynolds number in different geometries

(i) Reynold's experiment (1883)

Poiseuille flow in a circular tube
see https://www.youtube.com/watch?v=iY1YfWAIuBY

Re $\lessapprox$ Re_c	laminar flow
Re $\sim\mathrm{Re}_c$	turbulent 'surges' that travel with the flow
Re $\gtrapprox$ Re_c	turbulence - transverse mixing of the fluid

(ii) Flow over a cylinder

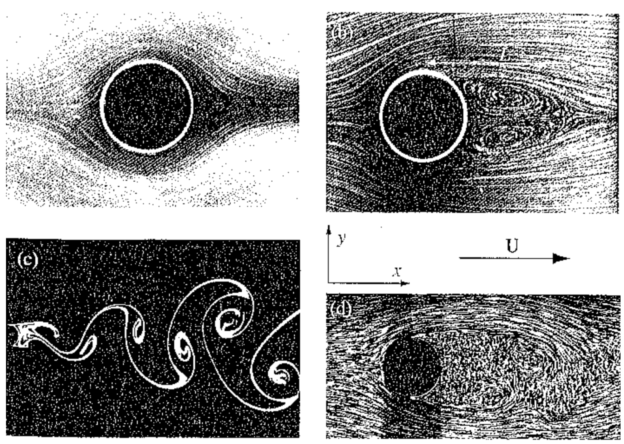


Figure 1: Flow over a cylinder for different Reynolds numbers. (a) Re=0.16, almost symmetric, (b) Re=26, eddies behind cylinder, fixed in space, (c) Re=200, Bénard - von Karman vortex street, (d) Re=8000, turbulent wake.

(d) checking it all works for Poiseuille flow

In Section A.8 we showed that the velocity field for pressure driven flow between two parallel plates is

$$u_x = \frac{p_0}{2\eta L} y(y-a). \tag{8}$$

For a given forcing, the solution should only depend on the Reynolds number. Let's check that this is the case. Define dimensionless variables

$$\tilde{u}_x = \frac{u_x}{U}, \qquad \tilde{y} = \frac{y}{a}$$

where U is a velocity scale and a is a length scale. It follows that

$$\tilde{L} = \frac{L}{a}, \qquad \frac{\tilde{p}_0}{\tilde{\rho}} = \frac{p_0}{U^2\rho}.$$

Writing the velocity profile (8) in terms of the dimensionless variables

$$U\tilde{u}_x = \frac{U^2\tilde{p}_0 a\tilde{y}(a\tilde{y}-a)}{2\tilde{\rho}\nu\tilde{L}a} = \frac{U^2\tilde{p}_0 a\tilde{y}(\tilde{y}-1)}{2\tilde{\rho}\nu\tilde{L}}.$$

Therefore

$$\tilde{u_x} = \frac{Ua}{\nu}\ \frac{\tilde{p_0}}{\tilde{\rho}\tilde{L}}\ \frac{\tilde{y}(\tilde{y}-1)}{2} \equiv Re\ \frac{\tilde{p_0}}{\tilde{\rho}\tilde{L}}\ \frac{\tilde{y}(\tilde{y}-1)}{2}.$$

$\tilde{p_0}/\tilde{\rho}\tilde{L}$ is the dimensionless acceleration. So, for a given forcing, the solution only depends on Re.

(e) Dynamical similarity

Flows with the same Re are identical if

- all geometrical features of the flow are scaled in the same way
- applied forces / pressure gradients are scaled appropriately
- there is no physics beyond N-S – or other dimensionless variables are needed eg:
  for compressible hydrodynamics, the Mach number, Ma=flow velocity/speed of sound.
  for a bouncing drop, the Weber number, We=kinetic energy/surface tension energy.

A.10 Vorticity

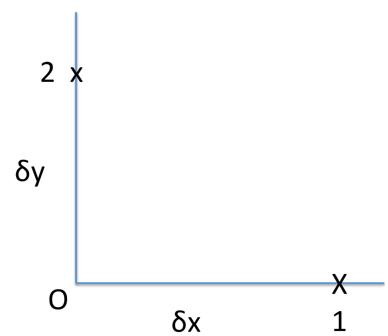
(a) definition and physical interpretation

$$\text{vorticity} \quad \boxed{\boldsymbol{\omega} = \operatorname{curl}\mathbf{u}}$$

Vorticity is useful, particularly at high Re, because of conservation theorems which we will come to later.

An *irrotational fluid* or region of fluid has zero vorticity.

Vorticity is a *local* measure of the spin of a fluid. In 2D vorticity $= 2 \times$ (average angular velocity of two infinitesimal, mutually perpendicular, fluid elements). To show this:



If the velocity at the origin is (u_x, u_y) then, Taylor expanding, the velocity at point 1 is

$$\left(u_x + \frac{\partial u_x}{\partial x}\,\delta x, u_y + \frac{\partial u_y}{\partial x}\,\delta x\right)$$

and the velocity at point 2 is

$$\left(u_x + \frac{\partial u_x}{\partial y}\,\delta y, u_y + \frac{\partial u_y}{\partial y}\,\delta y\right).$$

So the angular velocity of point 1 about O is $\frac{\partial u_y}{\partial x}$ and the angular velocity of point 2 about O is $-\frac{\partial u_x}{\partial y}$. So the average angular velocity of points 1 and 2 about O is

$$\frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\,\mathrm{curl}\,\mathbf{u} = \frac{1}{2}\boldsymbol{\omega}.$$

$\boldsymbol{\omega}$ can be measured by putting a tiny 'vorticity-meter' in the flow (Figure 2).

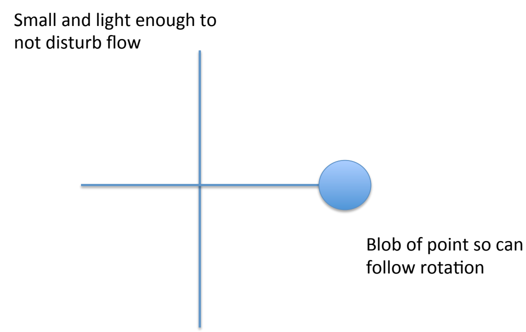


Figure 2: A 'vorticity meter' – a device to measure the vorticity. It has to be tiny because vorticity is a local property of the fluid.

(b) vortices
(ie things that look like whirlpools)

Consider the flow field

$$\mathbf{u} = \Omega f(r)\hat{\boldsymbol{\theta}} \tag{9}$$

where Ω is a constant that sets the velocity magnitude and we are using cylindrical polar co-ordinates (Fig. 3a).
The vorticity is

$$\boldsymbol{\omega} = \text{curl}\,\mathbf{u} = \quad r^{-1}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & \Omega r f(r) & 0 \end{vmatrix} = \frac{\Omega}{r}\frac{\partial}{\partial r}\{rf(r)\}\hat{\mathbf{z}}$$

which depends on $f(r)$.

For example, if $f(r) = r^{-1}$, then $\boldsymbol{\omega} = 0$. (The vorticity meter will not turn as it moves around the vortex core (Fig. 3b).) This is a good approximation of a bath plughole. However the zero vorticity condition must break down at $r = 0$ (mathematically) and in a small region around $r = 0$ (physically). A model for how this can happen is the Rankine vortex (see problem set).

3D versions of the same physics are vortex tubes and smoke rings which are stable structures at high Re (Fig. 3d).

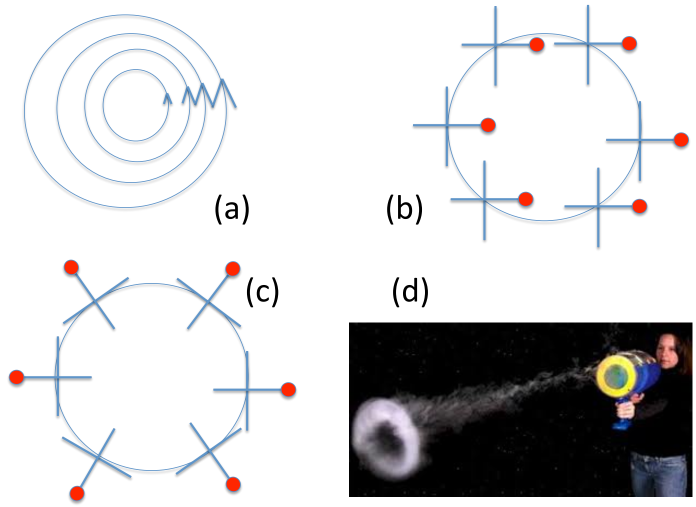


Figure 3: (a) a vortex: the flow field of Eq. (9), (b) a vortex with zero vorticity, (c) rigid body rotation, (d) a 3D vortex ring.

If $f(r) = r$, $\mathbf{u}$ describes rigid body rotation, with angular velocity Ω (Fig. 3c),

$$\boldsymbol{\omega} = \frac{\Omega}{r}\frac{\partial}{\partial r}\{r^2\}\hat{\mathbf{z}} = 2\Omega\hat{\mathbf{z}}.$$

(c) the vorticity equation

We will use the vector identities

$$\mathrm{curl}\,\mathrm{grad} \equiv 0, \tag{10}$$

$$\mathrm{curl}(\nabla^2\mathbf{u}) = \nabla^2(\mathrm{curl}\,\mathbf{u}), \tag{11}$$

$$\mathrm{curl}\{(\mathbf{u}\cdot\nabla)\mathbf{u}\} = (\mathbf{u}\cdot\nabla)\,\mathrm{curl}\,\mathbf{u} - (\mathrm{curl}\,\mathbf{u}\cdot\nabla)\mathbf{u} + (\nabla\cdot\mathbf{u})\,\mathrm{curl}\,\mathbf{u}. \tag{12}$$

The last term on the rhs of Eq. (12) is zero for an incompressible fluid.

Starting from the N-S equation

$$\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{\nabla p}{\rho} + \nu\nabla^2\mathbf{u},$$

taking the curl, and using Eqs. (10), (11), and (12),

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} = \nu\nabla^2\boldsymbol{\omega}$$

or, equivalently

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + \nu \nabla^2 \boldsymbol{\omega}.}$$

In the same way as we defined streamlines and streamtubes for the velocity:

a *vortex line* is a line that is everywhere tangent to $\boldsymbol{\omega}$.

a *vortex tube* is a bundle of vortex lines.