

YEAR 2: ELECTRICITY AND MAGNETISM

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PROBLEM SET 3: ELECTROMAGNETIC WAVES I

* Standard bookwork, included because you need to understand and learn this material, but your tutor may not want it handed in.

A. Maxwell's Equations

1. Write down Maxwell's equations.

How do they simplify for

(a) a linear, homogeneous, isotropic dielectric

(Explain what each term means – from now on we will assume a material is linear, homogeneous and isotropic unless explicitly stated otherwise.)

(b) a conductor,

(c) a good conductor?

B. Electromagnetic Waves in Dielectrics

2.* For a linear, homogeneous, isotropic dielectric of relative permittivity ϵ and relative permeability μ

(a) show that the electric (and magnetic) fields obey the wave equation and find the velocity of electromagnetic waves and the refractive index of the dielectric.

(b) show that plane electromagnetic waves are transverse.

(c) show that there are two linearly independent plane wave solutions, in each of which the electric field is perpendicular to the magnetic field.

(d) define and obtain an expression for the intrinsic impedance of the dielectric.

3. A non-magnetic, anisotropic, homogeneous dielectric (eg calcite) has constitutive relations

$$D_x = \epsilon_1 \epsilon_0 E_x, \quad D_y = \epsilon_1 \epsilon_0 E_y, \quad D_z = \epsilon_2 \epsilon_0 E_z.$$

(a) Show that waves propagate along z with a single speed.

(b) Specify the constraints on the orientation of the electric field that enable plane waves to propagate along y and show that waves of different polarization have different speeds.

C. Electromagnetic Waves in Conductors.

4.* (a) Show that, in a medium of conductivity σ , relative permittivity ϵ , and relative permeability μ , the field components of a plane wave are propagated as $e^{-\kappa z} e^{j(\omega t - kz)}$ where

$$k = \omega \sqrt{\frac{\epsilon \epsilon_0 \mu \mu_0}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \epsilon_0 \omega} \right)^2} + 1 \right)^{1/2},$$

$$\kappa = \omega \sqrt{\frac{\epsilon \epsilon_0 \mu \mu_0}{2}} \left(\sqrt{1 + \left(\frac{\sigma}{\epsilon \epsilon_0 \omega} \right)^2} - 1 \right)^{1/2}.$$

(b) Show that the (complex) intrinsic impedance of the conductor is

$$Z = \frac{\omega \mu \mu_0 e^{j\phi}}{\sqrt{k^2 + \kappa^2}}$$

where $\tan \phi = \kappa/k$.

5.(a) Distinguish between a good conductor and a poor conductor. Estimate the crossover frequency between the two regimes for (i) pure water, (ii) copper.

(b) Show that the skin depth in a poor conductor is $(2/\sigma) \sqrt{\epsilon \epsilon_0 / \mu \mu_0}$, independent of frequency. Find the skin depth of pure water.

(c) Show that the skin depth in a good conductor is $\lambda/2\pi$ where λ is the wavelength in the conductor. Find the skin depth for copper in the visible ($\omega \approx 10^{15} \text{s}^{-1}$).

(d) Find the phase difference between the magnetic and electric fields in copper, and calculate the ratio of their amplitudes in the visible.

(conductivities: copper $6 \times 10^7 \Omega^{-1} \text{m}^{-1}$, pure water $5 \times 10^{-6} \Omega^{-1} \text{m}^{-1}$; static relative permittivity of pure water $\epsilon = 80$, at optical frequencies this has dropped to $\epsilon = 1.8$)

6. A plane electromagnetic wave of frequency 10^{10} Hz passes through a non-magnetic, conducting medium. As the wave progresses through the medium its amplitude is halved each cm and the phase difference between the \vec{E} and \vec{H} vectors is 5° . Find the electrical conductivity and the relative permittivity of the medium.

D. Poynting Vector

7.* A region of space with volume V within a closed surface S contains electromagnetic fields of energy density u and a free current density \vec{J}_f . Applying conservation of energy to the volume V gives

$$\int_V \left(\frac{\partial u}{\partial t} + \nabla \cdot \vec{N} + \vec{J}_f \cdot \vec{E} \right) dV = 0.$$

Explain the origin of each term and use this equation, together with Maxwell's equations, to obtain expressions for u and the Poynting vector \vec{N} in terms of the electromagnetic fields \vec{E} , \vec{D} , \vec{B} , and \vec{H} .

8. Calculate the order of magnitude of the amplitudes of the electric and magnetic fields (in vacuum) in the following:

- (a) sunlight, which has an intensity at the earth $\sim 1 \text{ kWm}^{-2}$.
- (b) a laser beam in the visible which is focussed to a spot of diameter 10^{-2} mm and has an instantaneous power of 100MW.

9. A straight conducting wire of circular cross section and radius a has resistance R and carries a current I . Find the value and direction of the Poynting vector at a distance r from the centre of the wire. Hence show that:

- (a) the total power crossing the surface of the wire is $I^2 R$.
- (b) the energy is dissipated uniformly throughout the volume of the wire.

10. A long solenoid of n turns per unit length is wound upon a cylindrical core of radius a and relative permeability μ . The current I through the solenoid is increasing with time t at a constant rate. Obtain expression for the rate of increase of stored energy per unit length in the core of the solenoid

- (a) from the inductance per unit length of the solenoid, and dI/dt .
- (b) from the energy associated with the fields internal to the solenoid core.

(c) by integration of the Poynting vector over an appropriate surface.

11. (adapted from the University of Cambridge EM problem set)

A microwave oven operates at a frequency of 2.5GHz. A series of spherical potatoes of diameters 2.5, 5, 7.5 and 10cm are cooked individually in the oven until the surface is soft to the touch. The two smallest potatoes are cooked all the way through, but the larger two potatoes have a hard interior. Estimate the skin depth of potatoes at this frequency. Are potatoes good conductors or good insulators? For the largest potato, the oven delivers 600W of power. Estimate the average rms electric field in the oven.

12. Stationary spherical dust grains within the Solar System which have a radius less than r_c will be repelled by the Sun, whilst those with a radius greater than r_c will be attracted towards it. Explain why, and calculate r_c , given that the mass of the Sun is 2×10^{30} kg, its luminosity is 4×10^{26} W and the density of the dust grains is 10^3 kgm⁻³. (Neglect the pressure of the Solar Wind.)

E. Frequency dependence of the dielectric constant

13.(a) Use a simple, classical model of a dilute, non-polar gas to show that the complex refractive index \mathbf{n} is given by

$$\mathbf{n}^2 - 1 = \frac{n_0 e^2}{\epsilon_0 m} \frac{1}{(\omega_0^2 - \omega^2) - j\gamma\omega}$$

and explain the meaning of the symbols.

(b) Sketch the approximate dependence of the real and imaginary part of the refractive index on the angular frequency of the radiation incident on the medium.

(c) Explain why it is that in a dielectric medium

(i) the relative permittivity measured by a d.c. method is greater than unity.

(ii) the phase velocity of X-rays is greater than the velocity of light in free space.