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MULTIPLE INTEGRALS

Hilary Term — Prof J M Yeomans

1. Repeated integrals

(a) For the following integrals sketch the region of integration and so write equivalent integrals with the order of integration reversed. Evaluate the integrals both ways.

$$\int_0^{\sqrt{2}} \int_{y^2}^{2} y \, dx \, dy \ , \qquad \int_0^4 \int_0^{\sqrt{x}} y\sqrt{x} \, dy \, dx \ , \qquad \int_0^1 \int_{-y}^{y^2} x \, dx \, dy \ .$$

(b) Reverse the order of integration and hence evaluate:

$$\int_0^{\pi} \int_y^{\pi} x^{-1} \sin x \, dx \, dy \ .$$

2. Plane polar coordinates and planar mass distributions

(a) A mass distribution in the positive x region of the xy-plane and in the shape of a semi-circle of radius a, centred on the origin, has mass per unit area k. Find, using plane polar coordinates,

(i) its mass M, (ii) the coordinates $(\overline{x}, \overline{y})$ of its centre of mass, (iii) its moments of inertia about the x and y axes.

(b) Do as above for a semi-infinite sheet with mass per unit area

$$\sigma = k \exp -(x^2 + y^2)/a^2 \quad \text{for} \quad x \geq 0 \ , \qquad \sigma = 0 \quad \text{for} \quad x < 0 \ .$$

where a is a constant. Comment on the comparisons between the two sets of answers.

Note that

$$\int_0^{\infty} \exp\left(-\lambda u^2\right) du = \frac{1}{2}\sqrt{\frac{\pi}{\lambda}} \ .$$

(c) Evaluate the following integral:

$$\int_0^a \int_0^{\sqrt{a^2-y^2}} \left(x^2 + y^2\right) \arctan(y/x) \, dx \, dy \ .$$

3. **Jacobian matrix**

The pair of variables (x, y) are each functions of the pair of variables (u, v) and *vice versa*. Consider the matrices

$$A = \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

(a) Show using the chain rule that the product AB of these two matrices equals the unit matrix I.

(b) Verify this property explicitly for the case in which (x, y) are Cartesian coordinates and u and v are the polar coordinates (r, θ).

(c) Assuming the result that the determinant of a matrix and the determinant of its inverse are reciprocals, deduce the relation between the Jacobians

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{and} \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

4. **Change of variable in double integral**

(a) Using the change of variable $x + y = u$, $x - y = v$ evaluate the double integral $\iint_R (x^2 + y^2)\, dxdy$, where R is the region bounded by the straight lines $y = x$, $y = x + 2$, $y = -x$ and $y = -x + 2$.

(b) Given that $u = xy$ and $v = y/x$, show that $\partial(u,v)/\partial(x,y) = 2y/x$. Hence evaluate the integral

$$\iint \exp(-xy)\, dxdy$$

over the region $x > 0$, $y > 0$, $xy < 1$, $1/2 < y/x < 2$.

5. **Jacobian for change from Cartesian to spherical polar coordinates**

(This is an alternative derivation of the volume element in spherical polars.)

Spherical polar coordinates are defined in the usual way. Show that

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2 \sin\theta \ .$$

6. **Triple integrals over a hemisphere**

A solid hemisphere of uniform density k occupies the volume $x^2+y^2+z^2 \leq a^2$, $z \geq 0$. Using symmetry arguments wherever possible, find

(i) its total mass M, (ii) the position $(\overline{x}, \overline{y}, \overline{z})$ of its centre-of-mass, and (iii) its moments and products of inertia, $I_{xx},\ \ I_{yy},\ \ I_{zz},\ \ I_{xy},\ \ I_{yz},\ \ I_{zx}$, where

$$I_{zz} = \int k\ (x^2+y^2)\ dV, \qquad I_{xy} = \int k\ xy\ dV, \qquad \text{etc.}$$

7. **Surface area of a hemisphere**

Show that the surface area of the curved portion of the hemisphere in question (6) is $2\pi a^2$ by

(i) directly integrating the element of area $a^2 \sin\theta d\theta d\phi$ over the surface of the hemisphere.

(ii) projecting onto an integral taken over the x-y plane.

8. **Surface integrals**

(a) Find the area of the plane $x - 2y + 5z = 13$ cut out by the cylinder $x^2 + y^2 = 9$.

(b) A uniform lamina is made of that part of the plane $x + y + z = 1$ which lies in the first octant. Find by integration its area and also its centre of mass. Use geometrical arguments to check your result for the area.

Answers

1. (a) 1, 32/5, -1/15; (b) 2.

2. (a) $\pi k a^2/2$, $(4a/3\pi, 0)$, $Ma^2/4$, $Ma^2/4$; (b) $\pi k a^2/2$, $(a/\sqrt{\pi}, 0)$, $Ma^2/2$, $Ma^2/2$; (c) $\pi^2 a^4/32$.

4. (a) 8/3; (b) $(1 - e^{-1}) \ln 2$.

6. (i) $2\pi a^3 k/3$, (ii) $(0, 0, 3a/8)$, (iii) $2Ma^2/5$ for all moments of inertia and 0 for all products of inertia.

8. (a) $9\pi\sqrt{6/5}$; (b) $\sqrt{3}/2$, $(1/3, 1/3, 1/3)$.