COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

PROBLEM SET 3

Julia Yeomans

1. Solve the following differential equations ($y'' = \frac{d^2y}{dx^2}, y' = \frac{dy}{dx}$, etc.)

(i) $y'' + 2y' - 15y = 0$,

(ii) $y'' - 6y' + 9y = 0$, $\qquad y = 0, y' = 1$ at $x = 0$

(iii) $y'' - 4y' + 13y = 0$,
write the solution in terms of complex exponentials *and* in terms of sin and cos.

(iv) $y'' + k^2 y = 0$,
write the general solution in terms of complex exponentials *and* in terms of sin and cos. Is it possible to find a solution with $y = 0$ at $x = 0$ and $x = L$? For which values of k?

(v) $y''' + 7y'' + 7y' - 15y = 0$

2. A damped harmonic oscillator is displaced by a distance x_0 and released at time $t = 0$. Show that the subsequent motion is described by the differential equation

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = 0 \text{ with } x = x_0, \frac{dx}{dt} = 0 \text{ at } t = 0,$$

explaining the physical meaning of the parameters m, γ and ω_0.

(a) Find and sketch solutions for (i) overdamping, (ii) critical damping, and (iii) underdamping.
(iv) What happens for $\gamma = 0$?

(b) For a lightly damped oscillator the quality factor, or Q-factor, is defined as

$$Q = \frac{\text{energy stored}}{\text{energy lost per radian of oscillation}}.$$

Show that $Q = \omega_0/\gamma$.

3. Consider the differential equation

$$y'' - 3y' + 2y = f(x)$$

What would you try for the particular integral if $f(x) =$

(i) x^2	(ii) e^{4x}	(iii) e^x
(iv) $\sinh x$	(v) $\sin x$	(vi) $x \sin x$
(vii) $e^{2x} + \cos^2 x$		

4. Solve the following differential equations

(i)	$5y'' + 2y' + y = 2x + 3,$	$y = -1, y' = 0$ when $x = 0,$
(ii)	$y'' - y' - 2y = e^{2x},$	
(iii)	$4y'' - 4y' + y = 8e^{\frac{x}{2}},$	$y = 0, y' = 1$ when $x = 0,$
(iv)	$y'' + 3y' + 2y = xe^{-x},$	
(v)	$x'' + 4x = t + \cos 2t,$	$x = 0$ when $t = 0,$
(vi)	$y'' - 2y' + 2y = e^x(1 + \sin x),$	$y = 0$ when $x = 0, \frac{1}{2}\pi,$
(vii)	$1 + yy'' + (y')^2 = 0,$	
(viii)	$x^2y'' + xy' + y = x$	

5. Consider the damped oscillator of question 2 subject to an oscillatory driving force:

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = F\cos\omega t.$$

(i) Explain what is meant by the steady state solution of this equation, and calculate the steady state solution for the displacement $x(t)$ and the velocity $\dot{x}(t)$.
(ii) Sketch the amplitude and phase of $x(t)$ and $\dot{x}(t)$ as a function of ω.
(iii) Determine the resonant frequency for both the displacement and the velocity.
(iv) Defining $\Delta\omega$ as the full width at half maximum of the resonance peak (NB this is not the same definition as in the lectures) calculate $\Delta\omega/\omega_0$ to leading order in γ/ω_0.
(v) For a lightly damped, driven oscillator near resonance, calculate the energy stored and the power supplied to the system. Hence confirm that $Q = \omega_0/\gamma$ as in question 2. How is Q related to the width of the resonance peak?

6. Solve the coupled differential equations

$$\frac{dx}{dt} + ax - by = f,$$
$$\frac{dy}{dt} + ay - bx = 0,$$

where a, b, f are constants, for (i) $f = 0$, (ii) $f \neq 0$.

7. Solve the coupled differential equations

$$\frac{dy}{dx} + 2\frac{dz}{dx} + 4y + 10z - 2 = 0,$$
$$\frac{dy}{dx} + \frac{dz}{dx} + y - z + 3 = 0,$$

where $y = 0$ and $z = -2$ when $x = 0$.

Answers

1. (i) $y = Ae^{3x} + Be^{-5x}$
   (ii) $y = xe^{3x}$
   (iii) $y = e^{2x}(A\cos 3x + B\sin 3x) = e^{2x}(Ce^{3ix} + De^{-3ix})$
   (iv) $y = A\cos kx + B\sin kx = Ce^{ikx} + De^{-ikx}$,
   $y = B\sin\frac{n\pi x}{L}, n$ integer
   (v) $y = Ae^{x} + Be^{-3x} + Ce^{-5x}$

2. (i) $\gamma > 2\omega_0,\ x = \frac{x_0}{2\alpha}e^{-\frac{\gamma}{2}t}\{(\alpha + \frac{\gamma}{2})e^{\alpha t} + (\alpha - \frac{\gamma}{2})e^{-\alpha t}\}$ where $\alpha = \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{\frac{1}{2}}$

   (ii) $\gamma = 2\omega_0,\ x = x_0e^{-\frac{\gamma}{2}t}(1 + \frac{\gamma t}{2})$

   (iii) $\gamma < 2\omega_0,\ x = x_0e^{-\frac{\gamma}{2}t}(\cos\beta t + \frac{\gamma}{2\beta}\sin\beta t)$, where $\beta = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{\frac{1}{2}}$

   (iv) $\gamma = 0,\ \ x = x_0\cos\omega_0 t$ ie simple harmonic motion

3. (i) $Ax^2 + Bx + C$
   (ii) Ae^{4x}
   (iii) Axe^{x}
   (iv) $Axe^{x} + Be^{-x}$
   (v) $A\sin x + B\cos x$
   (vi) $(Ax + B)\sin x + (Cx + D)\cos x$
   (vii) $Axe^{2x} + B\cos 2x + C\sin 2x + D$

4. (i) $y = 2x - 5e^{-x/5}\sin\frac{2x}{5} - 1$
   (ii) $y = Ae^{-x} + Be^{2x} + \frac{1}{3}xe^{2x}$
   (iii) $y = (x + x^2)e^{\frac{x}{2}}$
   (iv) $y = Ae^{-x} + Be^{-2x} + \frac{1}{2}(x^2 - 2x)e^{-x}$
   (v) $x = \frac{1}{4}t(1 + \sin 2t) + A\sin 2t$
   (vi) $y = e^{x}(1 - \cos x - \sin x) - \frac{1}{2}x\cos xe^{x}$
   (vii) $(x + A)^2 + y^2 = B$
   (viii) $y = A\cos(\ln x) + B\sin(\ln x) + \frac{1}{2}x$

5. (i) $x = \frac{F\cos(\omega t - \phi)}{m\{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\}^{1/2}}, \quad \dot{x} = \frac{-\omega F\sin(\omega t - \phi)}{m\{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2\}^{1/2}}, \quad \tan\phi = \frac{\omega\gamma}{(\omega_0^2 - \omega^2)}$

   (iii) maximum of x is at $\omega^2 = \omega_0^2 - \frac{\gamma^2}{2}$, maximum of $\dot{x}$ is at $\omega = \omega_0$

   (iv) $\frac{\Delta\omega}{\omega_0} = \frac{\sqrt{3}\gamma}{\omega_0}$

   (v) stored energy is $\frac{F^2}{2m\gamma^2}$, mean power is $\frac{F^2}{2m\gamma}$

6. $x = e^{-at}(Ae^{bt} + Be^{-bt}) + \frac{af}{a^2 - b^2}, \quad y = e^{-at}(Ae^{bt} - Be^{-bt}) + \frac{bf}{a^2 - b^2}$

7. $y = -\frac{18}{5}e^{-2x} + \frac{28}{5}e^{-7x} - 2, \quad z = \frac{6}{5}e^{-2x} - \frac{21}{5}e^{-7x} + 1$