Quantum Mechanics MT 2024: Problem Sheet 2

Commutators : Physical Significance

2.1 What does it mean to say that two operators commute? What is the significance of two observables having mutually commuting operators?

Given that the commutator $[P, Q] \neq 0$ for some observables P and Q, does it follow that for all $|\psi\rangle \neq 0$ we have $[P, Q]|\psi\rangle \neq 0$?

2.2 What does it mean for two operators to be *compatible*? Prove that two observables are compatible if and only if the commutator between the associated Hermitian operators vanishes.

2.3 Show that if there is a complete set of mutual eigenkets of the Hermitian operators $\hat{A}$ and $\hat{B}$, then $[\hat{A}, \hat{B}] = \hat{0}$. Explain the physical significance of this result.

2.4 Let A and B be two Hermitian operators. Prove that if $[A, B] = 0$ there exists a complete set of simultaneous eigenstates of the operators A and B, i.e.

$$A|u_j\rangle = a_j|u_j\rangle\ , \quad B|u_j\rangle = b_j|u_j\rangle, \tag{1}$$

and $\{|u_j\rangle\}$ form a basis of the LVS on which A and B act.

(Note: you are allowed to assume Linear Algebra results from the Vectors and Matrices course on diagonalising operators; you do not need to derive these from scratch)

Time dependence and the Schrödinger equation

2.5 Write down the time-independent (TISE) and the time-dependent (TDSE) Schrödinger equations. Is it necessary for the wavefunction of a system to satisfy the TDSE? Under what circumstances does the wavefunction of a system satisfy the TISE?

2.6 Why is the TDSE first-order in time, rather than second-order like Newton's equations of motion?

2.7 A system has a time-independent Hamiltonian that has spectrum E_n. Prove that the probability P_k that a measurement of energy will yield the value E_k is is time-independent. Hint: you can do this either from Ehrenfest's theorem, or by differentiating $\langle E_k, t|\psi\rangle$ w.r.t. t and using the TDSE.

2.8 A particle is confined in a potential well such that its allowed energies are $E_n = n^2\mathcal{E}$, where $n = 1, 2, \ldots$ is an integer and $\mathcal{E}$ a positive constant. The corresponding energy eigenstates are $|1\rangle$, $|2\rangle$, ..., $|n\rangle, \ldots$ At $t = 0$ the particle is in the state

$$|\psi(0)\rangle = 0.2|1\rangle + 0.3|2\rangle + 0.4|3\rangle + 0.843|4\rangle.$$

(a) What is the probability, if the energy is measured at $t = 0$, of finding a number smaller than $6\mathcal{E}$?

(b) What is the mean value and what is the rms deviation of the energy of the particle in the state $|\psi(0)\rangle$?

(c) Calculate the state vector $|\psi\rangle$ at time t. Do the results found in (a) and (b) for time t remain valid for arbitrary time t?

(d) When the energy is measured it turns out to be $16\mathcal{E}$. After the measurement, what is the state of the system? What result is obtained if the energy is measured again?

Wavefunctions

2.9 Show that in one dimension, for functions which tend to zero as $|x| \to \infty$, the operator $\partial/\partial x$ is not Hermitian, but $-\mathrm{i}\hbar\partial/\partial x$ is. Is $\partial^2/\partial x^2$ Hermitian? What is the physical interpretation of the operator $-\mathrm{i}\hbar\partial/\partial x$?

2.10 Let $\psi(x, t)$ be the correctly normalized wave function of a particle of mass m and potential energy $V(x)$. Write down the expressions for the expectation values of (a) $\hat{x}$; (b) $\hat{x}^2$; (c) $\hat{p}_x$; (d) $\hat{p}_x^2$; (e) the energy.

What is the probability that the particle will be found in the interval (x_1, x_2)?

2.11 A particle moves in the potential $V(\boldsymbol{x})$ and is known to have energy E_n. (a) Can it have well-defined momentum for some particular $V(\boldsymbol{x})$? (b) Can the particle simultaneously have well-defined energy and position?

2.12 Consider a quantum mechanical particle with Hamiltonian

$$H = \frac{\hat{p}^2}{2m} + V(\hat{x}),$$

that is initially prepared in a state $|\psi(0)\rangle$. Using the TDSE show that the expectation value of an operator $\hat{Q}$ fulfils the following evolution equations

$$i\hbar \frac{d}{dt} \langle \psi(t)|\hat{Q}|\psi(t)\rangle = \langle \psi(t)|[\hat{Q}, H]|\psi(t)\rangle.$$

Consider the particular cases of the position and momentum operators and comment on the resulting equations