Introduction to Supersymmetry

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1 Introduction and References

These notes accompany the supersymmetry lecures given in Hilary and Trinity term 2009. There is nothing here that cannot be improved upon so here are some suggestions for further reading:

- *Weak Scale Supersymmetry*, CUP (2006), by Howard Baer and Xerxes Tata.

  The focus of this book is really supersymmetry at the LHC, written by two leading SUSY phenomenologists. If your interest in supersymmetry is primarily in the MSSM and its phenomenology, this is the book to buy.

- *The Soft Supersymmetry Breaking Lagrangian: Theory and Applications*, hep-ph/0312378, Chung et al.

  This is a good review of the MSSM and supersymmetry breaking. It's phenomenologically minded and is reasonably easy to read and dip in and out of. Not deep, but rather a broad first reference for TeV supersymmetry and its phenomenology.

- *Part III Supersymmetry Course Notes*, Fernando Quevedo and Oliver Schlotterer, http://www.damtp.cam.ac.uk/user/fq201/susynotes.pdf.

  These are the lecture notes for the 24 hour Part III supersymmetry course at Cambridge given by Fernando Quevedo, which this course is in part based upon. This should be at a similar level to the material presented here.

- *Supersymmetry and Supergravity*, Princeton University Press (1992), Julius Wess and Jonathan Bagger.

  Commonly known just as Wess and Bagger, this book is a genuine classic but is not dip-in reading. Dry but correct. Best used as a reference book. Very good on supergravity and has the 4d N=1 supergravity action in explicit complete gory detail.

- *Quantum Theory of Fields, Volume III*, CUP(2000), Steven Weinberg.

  Everything you ever wanted to know about supersymmetry. The advantage and disadvantage of these books is that Weinberg is one of the great masters of quantum field theory. You learn a lot, but you pay in blood for it.

- *Modern Supersymmetry*, OUP?(1995?), John Terning.

  A good general textbook on supersymmetry. A book with 'Modern' in its title dates itself by its topics. As a consequence this book is particularly strong on the aspects of non-perturbative supersymmetric field theory, but much weaker on supergravity.

2 Motivations for Supersymmetry

Why is supersymmetry even worth considering as a symmetry of nature? We present three reasons here, starting from the more formal and ending in the more phenomenological.

2.1 Susy is Special

We know nature likes symmetries. Symmetry principles underlie most of the dynamics of the Standard Model.

We are familiar with two basic types of symmetry. The particles of the Standard Model are defined as representations of the $SO(3,1)$ symmetry of Minkowski space. The Minwoski space metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{1}$$

is invariant under $SO(3,1)$ Lorentz transformations of the coordinates. These transformations are the Lorentz transformations and are the foundation of special relativity. For example, under a Lorentz transformation given by an $SO(3,1)$ matrix $M_{\mu\nu}$, a vector field A_μ transforms as

$$A_\mu^{new} = M_{\mu\nu} A_\nu^{old}. \tag{2}$$

Other representations are the spinor representation that describes fermions such as the quarks and leptons and the (trivial) scalar representation which describes the Higgs.

A second type of symmetry are internal symmetries. These may be either global symmetries (such as the approximate $SU(2)$ isospin symmetry of the strong interactions) or local (gauge) symmetries. An example of these is the $SU(3)$ gauge symmetry of the strong interactions which transforms quark states of different colour into one another. Internal symmetries do not change the Lorentz indices of a particle: an $SU(3)_c$ rotation changes the colour of a quark, but not its spin.

It seems like a cute idea to try and combine these two types of symmetries into one big symmetry group, which would contain spacetime $SO(3,1)$ in one part and internal symmetries such as $SU(3)$ in another. In fact this was an active area of research in prehistoric times (the 1960s), when there were attempts to combine all known symmetries into a single group such as $SU(6)$. However this program was killed by a series of no-go results, culminating in the Coleman-Mandula theorem (1967):

Coleman-Mandula Theorem: *The most general bosonic symmetry of scattering amplitudes (i.e. of the S-matrix) is a direct product of the Poincare and internal symmetries.*

$$G = G_{Poincare} \times G_{internal}$$

In other words, program over. There is no non-trivial way to combine spacetime and internal symmetries.

However, like all good no-go theorems, the Coleman-Mandula theorem has a loophole. Not knowing better, Coleman and Mandula only considered bosonic symmetry generators. In the early 1970s supersymmetry began to appear on the scene, and in 1975 Haag, Lopuskanski and Sohnius extended the Coleman-Mandula theorem to include the case of fermionic symmetry generators, which relate particles of different spins.

The result?

Haag, Lopuskanski and Sohnius: *The most general symmetry of the S-matrix is a direct product of super-Poincare and internal symmetries.*

$$G = G_{super-Poincare} \times G_{internal}$$

The super-Poincare algebra is the extension of the Poincare group to include transformations that turn bosons into fermions and fermions into bosons: *supersymmetry* transformations. So Coleman-Mandula were almost right, but not quite: there is one non-trivial way to extend the spacetime symmetries, and that is to incorporate supersymmetry.

The upshot of this is that supersymmetry appears in a special way: it is the unique extension of the Lorentz group as a symmetry of scattering amplitudes. This represents one reason to take supersymmetry seriously as a possible new symmetry of nature.

2.2 Strings and Quantum Gravity

Einstein's theory of general relativity is described, just like other theories, by a Lagrangian

$$\mathcal{L}_{GR} = 16\pi^2 M_P^2 \int d^4x \sqrt{g}\mathcal{R}.$$

Unlike the Standard Model, general relativity is a non-renormalisable theory: all interactions are suppressed by a scale $M_P = 2.4 \times 10^{18}\text{GeV} = 1.7 \times 10^{14} E_{LHC}$. This is somewhat analogous to the Fermi theory of weak interactions, which is also non-renormalisable, and where interactions are suppressed by $1/M_W^2 \sim 1/(100\text{GeV})^2$).

At energy scales below M_P, general relativity works just fine. However at energy scales above M_P, just as for Fermi theory above M_W, general relativity automatically loses predictive power. There are an infinite number of counterterms to be included in loop diagrams ($\mathcal{R}^2, \mathcal{R}^3, \ldots$) and no way to determine their coefficients. As all such higher-derivative operators become equally important above M_P, all predictivity is lost. This is the problem of quantum gravity.

Theorists rush where experimenters fear to tread, and despite (or in same cases because of) the absence of data lots of people have spent lots of time thinking about quantum gravity and how to formulate a consistent theory of it. The upshot is that the most attractive approach is that of string theory. String theory succeeds, often in surprising ways, in taming the divergences of quantum gravity and giving finite answers. It turns out supersymmetry plays a central and essential role in this: in any string theory, nature always looks supersymmetric at sufficiently high energy scales.

If string theory is telling us something about nature, nature is supersymmetric at some energy scale, giving us good reason to regard supersymmetry as a genuine symmetry of nature. The only problem is that it gives us no reason why this scale should be any smaller than the quantum gravity scale, $M_P = 2.4 \times 10^{18}$GeV.

2.3 The Hierarchy Problem

The hierarchy problem is the one really good reason why there is a fair chance supersymmetry will show up at the TeV energy scale.

The hierarchy problem is the problem of why the weak scale is so much smaller than the Planck scale. It is also the most serious theoretical problem of the Standard Model. To recall, in the Standard Model the electroweak symmetry is broken by a vacuum expectation value for the Higgs boson, $\langle v \rangle = 246$GeV. The size of this vev determines the masses of the Z and W bosons, $m_W = \frac{g_w}{\sqrt{2}} v$. The vev of the Higgs is determined by the parameters μ and λ in the Higgs potential,

$$V_H = -\mu^2 |\phi| + \lambda |\phi|^4. \quad (3)$$

As in any quantum theory, the parameters in this Lagrangian are subject to quantum corrections. The actual value of the Higgs vev is determined not by the classical values of μ and λ, but by those after quantum corrections. If the Standard Model is valid up to a scale Λ (i.e. no new particles appear until we reach the scale Λ) then we can estimate the size of quantum corrections to the term μ^2. The three largest contributions come from the Higgs self-loop, the W loop and the top quark loop. Let's focus on the top quark loop, shown in figure 2.3.

Figure 1: Quantum Corrections to the Higgs Mass.

This diagram gives a correction to the mass term in (3). This amplitude behaves as

$$\begin{aligned}
\mathcal{M} &\sim -i\int \frac{d^4k}{(2\pi)^4}\frac{-iy_t}{\not{k}-m_t}\frac{-iy_t}{\not{k}+\not{p}-m_t} \\
&\to_{|k|\to\infty} y_t^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2} \\
&\sim \frac{y_t^2}{16\pi^2}\int_{m_t}^{\Lambda}|k|dk \sim \frac{y_t^2}{16\pi^2}\Lambda^2. \qquad (4)
\end{aligned}$$

If the Standard Model is valid up to the GUT scale ($\Lambda \sim 10^{16}$GeV), the quantum correction to the Higgs mass term is of size $\delta\mu^2 \sim \frac{y_t^2}{16\pi^2}\Lambda^2 \gg m_W^2$. Quantum corrections to the Higgs potential are then enormous: we would expect the quantum-corrected Higgs vev to be around the GUT scale, not the weak scale. However, we know the masses of the W and Z bosons are $\mathcal{O}(10^2\text{GeV})$ and not $\mathcal{O}(10^{16})$GeV.

This is the hierarchy problem: why is the weak scale so much less than the Planck scale? There are two options. Either there is an enormous cancellation going on, or the Standard Model is not valid up to the GUT scale and new physics comes in at the TeV scale, cancelling the divergent terms in the Higgs potential.

TeV supersymmetry has the attractive feature of automatically taming the Higgs divergence. This finally makes it reasonable that supersymmetry is not just a symmetry of nature, but a symmetry that could appear at the next major collider, the LHC.

3 $SL(2,\mathbb{C})$ and dotted and undotted indices

Before we discuss the supersymmetry algebra it is helpful to establish notation and conventions. What we are doing here is setting up the appropriate language to write and manipulate 2-component spinors. The $SO(3,1)$ Lorentz algebra is

$$\begin{aligned}
[P^\mu, P^\nu] &= 0. \\
[M^{\mu\nu}, P^\alpha] &= i(P^\mu\eta^{\nu\alpha} - P^\nu\eta^{\mu\alpha}) \qquad (5)\\
[M^{\mu\nu}, M^{\rho\sigma}] &= i(M^{\nu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho}). \qquad (6)
\end{aligned}$$

Let's define $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$ where $i,j,k = 1,2,3$ as the generator of spatial rotations and $K_i = M_{0i}$ as the generator of Lorentz boosts. We can use these to rewrite the Lorentz algebra as

$$\begin{aligned}
[J_i, J_j] &= \epsilon_{ijk}J_k \qquad (7)\\
[K_i, K_j] &= -i\epsilon_{ijk}J_k \qquad (8)\\
[J_i, K_j] &= i\epsilon_{ijk}K_k. \qquad (9)
\end{aligned}$$

If we define

$$A_i = \frac{1}{2}(J_i + iK_i),$$

$$B_i = \frac{1}{2}(J_i - iK_i),$$

we can rewrite the Lorentz algebra as

$$
\begin{aligned}
[A_i, A_j] &= i\epsilon_{ijk} A_k & (10)\\
[B_i, B_j] &= i\epsilon_{ijk} B_k & (11)\\
[A_i, B_j] &= 0. & (12)
\end{aligned}
$$

This gives two copies of the $SU(2)$ algebra, from which we infer that, at least locally,

$$SO(3,1) \equiv SU(2) \times SU(2). \tag{13}$$

This tells us that we can label representations of $SO(3,1)$ by two $SU(2)$ 'charges': as $SU(2)$ reps are labelled by a positive integer $2S+1$, where S is the spin. Examples are

$$
\begin{aligned}
\text{Scalar} &\equiv (0,0)\\
\text{Left-handed spinor} &\equiv (\frac{1}{2}, 0)\\
\text{Right-handed spinor} &\equiv (0, \frac{1}{2})\\
\text{vector} &\equiv (\frac{1}{2}, \frac{1}{2}). & (14)
\end{aligned}
$$

Note that we should really replace $SO(3,1)$ by the universal covering group $Spin(3,1)$ to account for fermionic antiperiodicity.

For representing spinors it is helpful to use the map between $Spin(3,1)$ and $SL(2,\mathbb{C})$. $SL(2,\mathbb{C})$ is the group of *special* (unit determinant), *linear* (acts as a matrix) complex transformations on 2-component vectors. Briefly, left and right-handed spinors will be the fundamental and antifundamental representations of $SL(2,\mathbb{C})$.

Let's first describe the map between $Spin(3,1)$ and $SL(2,\mathbb{C})$. Suppose we have a 4d vector,

$$X = (x_0, x_1, x_2, x_3). \tag{15}$$

We can map this to a 2×2 hermitian matrix as

$$\tilde{X} \equiv X_\mu \sigma^\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \tag{16}$$

where $\sigma^\mu = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ are the standard Pauli matrices. Note that this map is invertible. Note that $\det \tilde{X} = X^\mu X_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

There are natural actions on both 4-component vectors and 2×2 matrices. On the vector the natural action is

$$X^\mu \to \Lambda^\mu_\nu X^\nu, \tag{17}$$

where Λ is an $SO(3,1)$ matrix. On the matrix the natural action is

$$\tilde{X} \to N \tilde{X} N^\dagger. \tag{18}$$

As the transformations are special they have unit determinant, and so $\det \tilde{X}' = \det \tilde{X}$. This implies that using the vector/matrix correspondence the transformations preserve the $SO(3,1)$ norm and thus correspond to Lorentz transformations. There is therefore a map between $SL(2,\mathbb{C})$ and $SO(3,1)$. You can check that the map is a homomorphism and preserves the group structure.

However, the map is not an isomorphism as

$$N = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ both map to } \mathbb{1} \in SO(3,1).$$

This reflects that the fact that $SL(2,\mathbb{C})$ is equivalent to $Spin(3,1)$ which is in fact the double cover of $SO(3,1)$.

- **Exercise:** Show that the explicit map from $SL(2,\mathbb{C}) \to SO(3,1)$ is given by

$$\Lambda^{\mu}_{\nu}(N) = \frac{1}{2}\mathrm{Tr}\left[\bar{\sigma}^{\mu} N \sigma_{\nu} N^{\dagger}\right]. \tag{19}$$

The relationship between $SL(2,\mathbb{C}) \equiv Spin(3,1)$ and $SO(3,1)$ is very similar to the relationship between $SU(2)$ and $SO(3)$: one is the double cover of the other.

3.1 Spinor Representations

The basic rule is that undotted indices correspond to left-handed $(\frac{1}{2}, 0)$ spinors and dotted indices correspond to right-handed $(0, \frac{1}{2})$ spinors.

The fundamental representation of $SL(2,\mathbb{C})$ corresponds to the left-handed spinor and the anti-fundamental representation to the right-handed spinor:

$$\psi'_{\alpha} = N_{\alpha}^{\beta} \psi_{\beta} \tag{20}$$

$$\chi'_{\dot{\alpha}} = (N^*)_{\dot{\alpha}}^{\dot{\beta}} \chi_{\dot{\beta}}. \tag{21}$$

Generally in susy formulae there are two kinds of indices: *spinor* $(\alpha, \beta, \dot{\alpha}, \dot{\beta})$ and *spacetime* (μ, ν, ρ, σ). Spacetime indices are raised and lowered by the metric tensor $g^{\mu\nu}$ and transform via $SO(3,1)$ matrices Λ^{μ}_{ν}.

Spinor indices are raised and lowered by the $SU(2)$ invariant tensor $\epsilon_{\alpha\beta}$, given by

$$\epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{22}$$

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{23}$$

$$\epsilon^{\alpha\beta}\epsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}. \tag{24}$$

Raised indices transform as

$$(\psi')^{\alpha} = \psi^{\beta} {N^{-1}}_{\beta}^{\ \alpha} \tag{25}$$

$$\chi^{\dot{\alpha}} = \chi^{\dot{\beta}} {(N^*)^{-1}}_{\dot{\beta}}^{\ \dot{\alpha}} \tag{26}$$

An example of a 'mixed' tensor is $\sigma^\mu_{\alpha\dot\alpha}$, which transforms as

$$(\sigma^\rho)_{\alpha\dot\alpha} = \underbrace{(\Lambda^{-1})^\rho_\nu}_{SO(3,1)} \underbrace{N^\beta_\alpha (\sigma^\nu)_{\beta\dot\beta} (N^*)^{\dot\beta}_{\dot\alpha}}_{SL(2,C)} \tag{27}$$

We can also define $(\bar\sigma^\mu)^{\dot\alpha\alpha} \equiv \epsilon^{\dot\alpha\dot\beta}\epsilon^{\alpha\beta}\sigma^\mu_{\beta\dot\beta}$. This has the standard form $\bar\sigma^\mu = (1, -\sigma^i)$. We can now define

$$(\sigma^{\mu\nu})^\beta_\alpha = \frac{i}{4}(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)^\beta_\alpha \tag{28}$$

$$(\bar\sigma^{\mu\nu})^{\dot\alpha}_{\dot\beta} = \frac{i}{4}(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu)^\beta_\alpha = (\sigma^{\mu\nu})^\dagger \tag{29}$$

These matrices satisfy the Lorentz algebra

$$[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] = i(\sigma^{\mu\sigma}\eta^{\nu\rho} + \sigma^{\nu\rho}\eta^{\mu\sigma} - \sigma^{\mu\rho}\eta^{\nu\sigma} - \sigma^{\nu\sigma}\eta^{\mu\rho}) \tag{30}$$

and we can use this to write general Lorentz transformations as

$$N^\beta_\alpha = \left(e^{-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}}\right)^\beta_\alpha. \tag{31}$$

Some useful identities are

$$\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{32}$$

$$\mathrm{Tr}(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu} \tag{33}$$

$$\sigma^\mu_{\alpha\dot\alpha}\bar\sigma^{\dot\beta\beta}_\mu = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} \tag{34}$$

$$\sigma^{\mu\nu} = \frac{1}{2i}\epsilon^{\mu\nu\rho\sigma}\sigma_{\rho\sigma} \tag{35}$$

$$\bar\sigma^{\mu\nu} = \frac{-1}{2i}\epsilon^{\mu\nu\rho\sigma}\bar\sigma_{\rho\sigma}. \tag{36}$$

Spinor products and Fierz identities are

$$\chi\psi \equiv \chi^\alpha\psi_\alpha = -\chi_\alpha\psi^\alpha = \psi^\alpha\chi_\alpha = \psi\chi. \tag{37}$$

$$\bar\chi\bar\psi \equiv \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = -\bar\chi^{\dot\alpha}\bar\psi_{\dot\alpha} = \bar\psi\bar\chi. \tag{38}$$

$$\psi\psi = \psi^\alpha\psi_\alpha = \epsilon^{\alpha\beta}\psi_\beta\psi_\alpha = \psi_2\psi_1 - \psi_1\psi_2. \tag{39}$$

For Grassmanian ψ, we can also write $\psi_\alpha\psi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\psi\psi$. We can also enumerate a useful set of Fierz identities:

$$(\theta\psi)(\theta\psi) = -\frac{1}{2}(\psi\psi)(\theta\theta) = -\frac{1}{2}(\theta\theta)(\psi\psi). \tag{40}$$

$$\psi\sigma^{\mu\nu}\chi = -\chi\sigma^{\mu\nu}\psi. \tag{41}$$

$$\psi_\alpha\chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\chi) + \frac{1}{2}\left(\sigma^{\mu\nu}\epsilon^T\right)_{\alpha\beta}(\psi\sigma_{\mu\nu}\chi). \tag{42}$$

$$\psi_\alpha\bar\chi_{\dot\alpha} = \frac{1}{2}(\psi\sigma_\mu\bar\chi)\sigma^\mu_{\alpha\dot\alpha}. \tag{43}$$

$$(\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = \chi_\alpha * \psi^{*,\alpha} = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi. \tag{44}$$

4 Supersymmetry Algebras

4.1 Form of the Supersymmetry Algebra

The supersymmetry algebra involves extending the ordinary Lorentz symmetry group of 4d quantum field theory by additional fermionic generators. The fermionic generators are Weyl spinors.

We include N such generators Q_α^A,$A = 1 \dots N$ into the algebra, as well as their Hermitian conjugates $\bar{Q}_{al\dot{p}ha}$. Here α and $\dot{\alpha}$ are both spinor indices, transforming as the fundamental(anti-fundamental) of $SU(2, \mathbb{C})$. Undotted indices correspond to left-handed fermions, and dotted indices correspond to right-handed fermions. A Weyl spinor has $2^{D/2}$ complex components, and so in 4 dimensions has 4 real components.

A supersymmetry algebra with 1 spinor generator Q_α is called $\mathcal{N} = 1$ supersymmetric, one with two spinor generators Q_α^1, Q_α^2 called $\mathcal{N} = 2$ supersymmetric, etc. As a Weyl spinor in 4d has 4 real components, a 4d theory with $\mathcal{N} = 1$ supersymmetry has a total of 4 real supercharges. Note that this is dimension-dependent. For example, in 10 dimensions the smallest spinor representation is the Majorana-Weyl representation, which has 8 complex components. $\mathcal{N} = 1$ supersymmetry in 10d therefore has sixteen supercharges, and $\mathcal{N} = 2$ supersymmetry thirty-two supercharges.

The $\mathcal{N} = 1$ supersymmetry algebra is

$$\left\{ Q_\alpha, \bar{Q}_{\dot{\beta}} \right\} = 2\sigma^m_{\alpha\dot{\beta}} P_m, \quad (45)$$

$$\{Q_\alpha, Q_\beta\} = \left\{ \bar{Q}_{\dot{\alpha}}, \bar{Q}_{\dot{\beta}} \right\} = 0, \quad (46)$$

$$[P_m, Q_\alpha] = \left[P_m, \bar{Q}_{\dot{\alpha}} \right] = 0, \quad (47)$$

$$[P_m, P_n] = 0. \quad (48)$$

As the susy generators are spinors, they naturally anticommute rather than commute. There are also transformations involving the Lorentz generators $\Lambda^{\alpha\beta}$, which are not shown here. These transformations can be summarised by the statement that Q_α transforms as a spinor under Lorentz transformations, so

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^\beta Q_\beta. \quad (49)$$

In general susy transformations commute with all internal symmetries (for example flavour symmetries). The exceptions to this are known as *R-symmetries* (R-parity in the MSSM is an example of an R-symmetry). Such symmetries arise from automorphisms of the susy algebra.

$$Q_\alpha \longrightarrow e^{i\gamma} Q_\alpha, \quad (50)$$

$$\bar{Q}_{\dot{\alpha}} \longrightarrow e^{-i\gamma} \bar{Q}_{\dot{\alpha}}. \quad (51)$$

If this is a symmetry of the Lagrangian (in general it isn't), then it represents a $U(1)$ R-symmetry. The generator R has commutation relations

$$[Q_\alpha, R] = Q_\alpha, \quad (52)$$

$$\left[\bar{Q}_{\dot{\alpha}}, R \right] = -\bar{Q}_{\dot{\alpha}}. \quad (53)$$

Note that R-symmetries act differently on bosons and fermions in the same supermultiplet (cf R parity in the MSSM, which has value 1 for Standard Model particles and -1 for their MSSM superpartners).

Suppose $R|b\rangle = r|b\rangle$, and $|f\rangle = Q|b\rangle$. Then

$$RQ|b\rangle - QR|b\rangle = Q|b\rangle, \tag{54}$$

and so $R(Q|b\rangle) = (r+1)(Q|b\rangle)$, giving

$$R|f\rangle = (r+1)|f\rangle, \text{ for } R|b\rangle = r|b\rangle. \tag{55}$$

It is relatively straightforward to extend the supersymmetry algebra to more generators. If we have N generators, the algebra becomes ($Q_\alpha \to Q^A_\alpha, A = 1 \dots N$)

$$\begin{aligned} \left\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\right\} &= 2\left(\sigma^\mu_{\alpha\dot\beta}P_\mu\right)\delta^A_B. \\ \left\{Q^A_\alpha, Q^B_\beta\right\} &= Z^{AB}. \end{aligned} \tag{56}$$

Z is called the *central charge* of the algebra and it commutes with all other generators.

$$\left[Z^{AB}, P^\mu\right] = \left[Z^{AB}, M^{\mu\nu}\right] = \left[Z^{AB}, Q^A_\alpha\right] = \left[Z^{AB}, Z^{CD}\right] = 0.$$

The antisymmetry of Z means it can only be present in algebras with extended ($N \geq 2$) supersymmetry. Z can be diagonalised. As it is antisymmetric, Hermitian transformations can be used to bring it to the canonical forms

$$\text{N even} \quad Z = \begin{pmatrix} 0 & q_1 & 0 & 0 & \dots & \dots & 0 & 0 \\ -q_1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & q_2 & \dots & \dots & 0 & 0 \\ 0 & 0 & q_2 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & q_{N/2} \\ 0 & 0 & 0 & 0 & \dots & \dots & -q_{N/2} & 0 \end{pmatrix} \tag{57}$$

$$\text{N odd} \quad Z = \begin{pmatrix} 0 & q_1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -q_1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & q_2 & \dots & 0 & 0 & 0 \\ 0 & 0 & q_2 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & q_{N/2} & 0 \\ 0 & 0 & 0 & 0 & \dots & -q_{N/2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{58}$$

4.2 Representations of the Supersymmetry Algebra

Given a symmetry group, the first important thing to do is to study the representations. For the Lorentz group, the study of the representations leads to the classification of particles as either scalar, spinor or vector. For the gauge symmetry groups $SU(3)$ and $SU(2)$, the study of the representations underlies the structure and interactions of the Standard Model. In a similar way the structure and interactions of supersymmetric theories are determined to a large extent by the representations of the susy algebra.

We start by proving an important if unsurprising result: the number of bosons and fermions in any massive representation is identical.

Consider the operator $(-1)^{N_F}$, which acts on 1-particle states as

$$(-1)^{N_F}|b\rangle = |b\rangle, \quad (59)$$
$$(-1)^{N_F}|f\rangle = -|f\rangle. \quad (60)$$

The generalisation to n-particle states is obvious. As Q_α converts fermions into bosons and vice-versa,

$$(-1)^{N_F} Q_\alpha = -Q_\alpha (-1)^{N_F}. \quad (61)$$

Given a finite-dimensional representation, we therefore have

$$\mathrm{Tr}\left[(-1)^{N_F}\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\}\right] = \mathrm{Tr}\left[(-1)^{N_F}\left(Q^A_\alpha \bar{Q}_{\dot{\beta}B} + \bar{Q}_{\dot{\beta}B} Q^A_\alpha\right)\right] \quad (62)$$
$$= \mathrm{Tr}\left[-Q^A_\alpha (-1)^{N_F} \bar{Q}_{\dot{\beta}B} + Q^A_\alpha (-1)^{N_F} \bar{Q}_{\dot{\beta}B}\right] \quad (63)$$
$$= 0. \quad (64)$$

However $\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = 2\left(\sigma^\mu_{\alpha\dot{\beta}} P_\mu\right)\delta^A_B$, and so the left-hand side becomes

$$\mathrm{TR}\left[(-1)^{N_F} 2\left(\sigma^\mu_{\alpha\dot{\beta}} P_\mu\right)\delta^A_B\right].$$

Q^A_α commutes with P_μ, and so all states in the representation have the same momentum. We can therefore pull the 4-vector $\langle P_\mu \rangle$ out, to get

$$0 = 2\langle P_\mu \rangle \sigma^\mu_{\alpha\dot{\beta}} \delta^A_B \mathrm{Tr}\left[(-1)^{N_F}\right]. \quad (65)$$

For this to hold for all α, β, A, B we need

$$\mathrm{Tr}\left[(-1)^{N_F}\right] = 0. \quad (66)$$

That is, the representation contains identical numbers of bosons and fermions.

Note that this result required $\langle P_\mu \rangle \neq 0$. This is not a mere technicality. For example, the vacuum of a quantum field theory has $\langle P_\mu \rangle = 0$, and for the vacuum $Q_\alpha |vac\rangle = 0$: there are only bosonic states in the representation. In the specific case of the vacuum $\mathrm{Tr}\left[(-1)^{N_F}\right]$ is known as the *Witten index*. Although currently this seems a bit trivial, we shall see later that we can use this to give important information about quantum field theories at strong coupling.

4.3 One-particle representations

Let us now consider one particle representations of the supersymmetry algebra. We will start with massless particles, and then move on to massive representations. Recall that the algebra is

$$\left\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\right\} = 2\left(\sigma^\mu_{\alpha\dot{\beta}} P_\mu\right)\delta^A_B.$$

For a massless particle we can boost to a light-like reference frame where the 4-momentum is given by $P_\mu = (E, 0, 0, E)$. In this case

$$\sigma^\mu_{\alpha\dot\beta} P_\mu = E\sigma^0 + E\sigma^3 \tag{67}$$
$$= E\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) \tag{68}$$
$$= 2E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta}. \tag{69}$$

The supersymmetry algebra is therefore given by

$$\left\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\right\} = 4E\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\beta} \delta^A_B. \tag{70}$$

Specialising to $\mathcal{N} = 1$ supersymmetry we obtain

$$\left\{Q_1, \bar{Q}_1\right\} = 4E, \tag{71}$$
$$\left\{Q_2, \bar{Q}_2\right\} = 0. \tag{72}$$

Consider a state $|P, \lambda\rangle$ of momentum P and helicity λ. If we write $a = \frac{Q_1}{2\sqrt{E}}$ and $a^\dagger = \frac{\bar{Q}_1}{2\sqrt{E}}$, then we obtain the algebra

$$\left\{a, a^\dagger\right\} = 1, \qquad \{a, a\} = \{a^\dagger, a^\dagger\} = 0.$$

This is precisely the algebra of fermionic creation and annihilation operators, suggesting that we can form representations by starting at a vacuum state $|\Omega\rangle$ and acting with creation operators. Furthermore, as $[Q_\alpha, J^3] = \frac{1}{2}\sigma^3_{\alpha\beta} Q_\beta$ (as Q_α transforms as a Weyl spinor), we have $[Q_1, J^3] = \frac{1}{2}Q_1$, and so $[a, J_3] = \frac{1}{2}a$. We then obtain

$$J_3 a|P, \lambda\rangle = (-[a, J_3] + aJ_3)\,|P, \lambda\rangle \tag{73}$$
$$= \left(-\frac{a}{2} + a\lambda\right)|P, \lambda\rangle \tag{74}$$
$$= a(\lambda - \frac{1}{2})|P, \lambda\rangle. \tag{75}$$

Therefore $a|P, \lambda\rangle$ has helicity $\lambda - \frac{1}{2}$ and $a^{dagger}|P, \lambda\rangle$ has helicity $(\lambda + \frac{1}{2})$. So creation operators increase the helicity of a state by $\frac{1}{2}$ and annihiliation operators decrease the helicity by $\frac{1}{2}$.

Also note that $Q_2|P, \lambda\rangle = 0$ for any state. Why?

$$\{Q_2, \bar{Q}_2\} = 0 \;\rightarrow\; \langle P, \lambda|Q_2\bar{Q}_2 + \bar{Q}_2 Q_2|P, \lambda\rangle = 0 \tag{76}$$
$$\rightarrow\; \langle P, \lambda|Q_2\bar{Q}_2|P, \lambda\rangle = -\langle P, \lambda|\bar{Q}_2 Q_2|P, \lambda\rangle \tag{77}$$
$$= \; ||Q_2|P, \lambda\rangle||^2 = -||\bar{Q}_2|P, \lambda\rangle||^2. \tag{78}$$

However positivity of the norm therefore requires $Q_2|P, \lambda\rangle = 0$ for all states.

We can now examine the representations of the susy algebra. We suppose we have a state $|\Omega\rangle$ in the representation annihilated by all annihilation operators. Then if $|Omega\rangle = |P, \lambda\rangle$, $a^\dagger|Omega\rangle = |P, \lambda + \frac{1}{2}\rangle$ and $a^\dagger a^\dagger|\Omega\rangle = 0$. If a theory is also invariant under CPT (this is not an algebraic consequence, but does hold for all relativistic quantum field theories) then each state must have a CPT conjugate. Normally the expression 'susy representation' will be used for the full set of states,

$$|\Omega\rangle + a^{\dagger}|Omega\rangle + |Omega'\rangle + a|\Omega^{'}\rangle >$$

where $|Omega' >$ is the CPT conjugate of $|Omega >$.

We can now enumerate the classic representations of $\mathcal{N} = 1$ supersymmetry.

Chiral Multiplet

This has $\lambda = 0$ and so $|\Omega\rangle = |P, 0\rangle$. The states are

$$|Omega\rangle = |P, 0\rangle, a^{\dagger}|\Omega\rangle = |P, \frac{1}{2}\rangle, |\Omega'\rangle = |P, 0 >', a|\Omega' >= |P, -\frac{1}{2} > .$$

The physical content of the chiral multiplet is one Weyl spinor and one complex scalar. The chiral multiplet is so called because it allows matter in chiral representations of the gauge group. Examples of chiral multiplets in the MSSM are $(\lambda = 0, \lambda = \frac{1}{2})$ being (squark, quark), (higgs, higgsino), (slepton, lepton).

Vector Multiplet

This has $\lambda = \frac{1}{2}$ and so $|\Omega\rangle = |P, \frac{1}{2}\rangle$. The states are

$$|Omega\rangle = |P, \frac{1}{2}\rangle, a^{\dagger}|\Omega\rangle = |P, 1\rangle, |\Omega^{'}\rangle = |P, -\frac{1}{2} >^{'}, a|\Omega^{'} >= |P, -1 > .$$

The physical content of the vector multiplet is one 2-component spinor (gaugino) and one vector boson (a gauge boson). The vector multiplet generalises gauge interactions to supersymmetric field theories. Examples of vector multiplets in the MSSM are $(\lambda = \frac{1}{2}, \lambda = 1)$ being (gluino, gluon), (photino, photon), (Wino, W).

Gravity Multiplet

This has $\lambda = 3/2$ and so $|\Omega\rangle = |P, 3/2\rangle$. The states are

$$|Omega\rangle = |P, 3/2\rangle, a^{\dagger}|\Omega\rangle = |P, 2\rangle, |\Omega^{'}\rangle = |P, -3/2 >^{'}, a|\Omega^{'} >= |P, -2 > .$$

The physical content of the gravity multiplet is one spin 3/2 vector-spinor (gravitino) and one spin 2 multiplet (the graviton). The gravity multiplet does not play a role in globally supersymmetric quantum field theories, but is essential to the formulation of locally supersymmetric theories (supergravity).

4.4 Massless Representations of $\mathcal{N} \geq 2$ supersymmetry

If $P^{\mu} = (E, 0, 0, E)$ the susy algebra was

$$\left\{ Q^{A}_{\alpha}, \bar{Q}_{\dot{\beta}B} \right\} = 4E \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_{\alpha\dot{\beta}} \delta^{A}_{B}.$$

So for $\mathcal{N} > 1$ susy we just have

$$\left\{Q_1^A, \bar{Q}_{1B}\right\} = 4E\delta_B^A,$$

$$\left\{Q_2^A, \bar{Q}_{2B}\right\} = 0.$$

This simply generalises the $N = 1$ algebra so that we now have N copies of fermionic creation and annihilation operators instead of just a single copy. As $[Q_1^A, J_3] = \frac{1}{2}Q_1^A$, we find

$$J_3 a^A |P, \lambda >= a^A (\lambda - \frac{1}{2}) |P, \lambda >$$

for all A, and so each lowering operator reduces helicity by $\frac{1}{2}$ and each raising operator increases helicity by $\frac{1}{2}$. The general state in a representation is therefore

$$\epsilon^{AB\ldots K} a_A^\dagger a_B^\dagger \ldots a_K^\dagger |\Omega >$$

where ϵ is antisymmetric and $0 \leq K \leq N$. There are $\left(\begin{array}{c} N \\ K \end{array}\right)$ states of helicity $\lambda + K/2$. The ground state $|\Omega\rangle$ has helicity λ and the top state $a_1^\dagger a_2^\dagger \ldots a_N^\dagger |\Omega\rangle$ helicity $\lambda + N/2$. What are the classic representations of $N \geq 2$ supersymmetry?

N=2 Hypermultiplet

This has $|\Omega\rangle = |P, -\frac{1}{2}\rangle$. It has the following states and helicities

$$a_1^\dagger a_2^\dagger |\Omega\rangle \qquad -\frac{1}{2} \tag{79}$$

$$a_1^\dagger |\Omega\rangle, a_2^\dagger |\Omega\rangle \qquad 0, 0 \tag{80}$$

$$|\Omega\rangle \qquad -\frac{1}{2} \tag{81}$$

This is the hypermultiplet. Important things to note are

1. As susy generators commute with gauge generators, the states of helicity $\pm\frac{1}{2}$ have the *same* gauge charges. This representation (and in fact all $N = 2$ representations) is therefore non-chiral.

2. If the gauge representation is real, the $N = 2$ hypermultiplet is CPT self-conjugate.

3. The particle fermion is one spin-(1/2) fermion and one complex scalar. Including a CPT conjugate, we have two spin-(1/2) fermions and two complex scalars.

N=2 Vector multiplet

This has $|\Omega\rangle = |P, 0\rangle$. It has the following states and helicities

$$a_1^\dagger a_2^\dagger|\Omega\rangle \qquad +1 \tag{82}$$
$$a_1^\dagger|\Omega\rangle, a_2^\dagger|\Omega\rangle \qquad +1/2, +1/2 \tag{83}$$
$$|\Omega\rangle, |\Omega'\rangle \qquad 0, 0 \tag{84}$$
$$a_1|\Omega'\rangle, a_2|\Omega'\rangle \qquad -1/2, -1/2 \tag{85}$$
$$a_1 a_2|\Omega'\rangle \qquad -1. \tag{86}$$

This is the *vector multiplet*, which contains a spin-1 gauge boson. The matter content of this multipet is one vector boson, one complex scalar and two spin-1/2 fermions.

N=4 Vector Multiplet

There is only one important representation for $N = 4$ supersymmetry, the vector multiplet. Here $|\Omega\rangle = |P, -1\rangle$. The states, helicities and multiplicities are

$$a_1^\dagger a_2^\dagger a_3^\dagger a_4^\dagger|\Omega\rangle \qquad +1 \qquad 1 \tag{87}$$
$$a_1^\dagger a_2^\dagger a_3^\dagger|\Omega\rangle, \qquad +1/2 \qquad 4 \tag{88}$$
$$a_1^\dagger a_2^\dagger|\Omega\rangle \qquad 0 \qquad 6 \tag{89}$$
$$a_1^\dagger|\Omega\rangle \qquad -1/2 \qquad 4 \tag{90}$$
$$|\Omega\rangle \qquad -1 \qquad 1. \tag{91}$$

This is the $\mathcal{N} = 4$ vector multiplet. It consists of one vector boson, four spin-1/2 fermions and three complex scalars. This is the multiplet present in the $\mathcal{N} = 4$ super Yang-Mills theory that has played such a prominent role in the AdS/CFT correspondence. $N = 4$ supersymmetry is maximal in global supersymmetry, as once we go above $N = 4$ we must necessarily include the gravitino in the spectrum, and this then also requires a graviton for a consistent theory.

N=8 Gravity Multiplet

This is essentially the unique representation of $\mathcal{N} = 8$ supersymmetry. $\mathcal{N} = 8$ supersymmetry is often called maximal supersymmetry, as there is no consistent known way of writing interacting theories for fundamental particles with spins greater than 2. The states are

$$\begin{array}{ccc} (a^\dagger)^8|\Omega\rangle & +2 & 1 \\ (a^\dagger)^7|\Omega\rangle, & +3/2 & 8 \\ (a^\dagger)^6|\Omega\rangle & +1 & 28 \\ (a^\dagger)^5|\Omega\rangle & +1/2 & 56 \\ (a^\dagger)^4|\Omega\rangle & 0 & 70 \\ (a^\dagger)^3|\Omega\rangle, & -1/2 & 56 \\ (a^\dagger)^2|\Omega\rangle & -1 & 28 \\ (a^\dagger)^1|\Omega\rangle & -3/2 & 8 \\ |\Omega\rangle & -2 & 1. \end{array} \qquad (92)-(100)$$

This is the gravity multiplet of $\mathcal{N} = 8$ supersymmetry. The states consist of one graviton, eight gravitini, 28 gauge bosons, 56 spin-1/2 fermions and 35 complex scalars.

With the N=8 gravity multiplet we conclude our study of massless representations of the supersymmetry algebra. We now move to massive representations.

4.5 Massive One-Particle Representations

For massive particles we can boost to a frame where the 4-momentum is $P^\mu = (M, 0, 0, 0)$. In this case the susy algebra looks like (N=1)

$$\left\{Q_\alpha, \bar{Q}_{\dot\beta}\right\} = 2\left(\sigma^\mu_{\alpha\dot\beta} P_\mu\right) = 2E \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Writing $\frac{Q_1}{\sqrt{2E}}, \frac{Q_2}{\sqrt{2E}}$ as a_1, a_2 and $\frac{\bar{Q}_{\dot 1}}{\sqrt{2E}}, \frac{\bar{Q}_{\dot 2}}{\sqrt{2E}}$ as $a_1^\dagger, a_2^\dagger$, we have

$$\{a_1, a_1^\dagger\} = 1, \qquad (101)$$

$$\{a_2, a_2^\dagger\} = 1. \qquad (102)$$

In contrast to the massless case we now have *two* sets of fermionic creation and annihilation operators. In general, for the case of N susy generators (without central charges) we have 2N operators

$$\{a_1^A, a_1^{\dagger,B}\} = \delta^{AB}, \qquad (103)$$

$$\{a_2^A, a_2^{\dagger,B}\} = \delta^{AB}. \qquad (104)$$

Representations can be constructed as before by starting with a ground state $|\Omega\rangle$ and acting with raising operators. However, there are some differences compared to the massless case. First, as $[Q, J_3] = \frac{1}{2}\sigma_3 Q$, we have

$$[a_1, J_3] = \frac{1}{2} a_1$$

$$[a_2, J_3] = -\frac{1}{2} a_2$$

This means that $a_1^\dagger$ increases the z spin by 1/2, while $a_2^\dagger$ decreases it by 1/2. Second, as $P^\mu = (M, 0, 0, 0)$ the little group is $SO(3)$ and particles in automatically in $SO(3) \approx SU(2)$ representations. So the total set of states in a massive N=1 susy representation is given by

$$a_1^\dagger a_2^\dagger |\Omega, J\rangle + SO(3) \text{ rotations} , \qquad (C)$$

$$a_1^\dagger |\Omega, J\rangle, a_2^\dagger |\Omega, J\rangle + SO(3) \text{ rotations} \qquad (B)$$

$$|\Omega, J\rangle + \text{SO(3) rotations} \qquad (A)$$

As we have already seen $\left(\ a_1^\dagger a_2^\dagger \ \right)$ transforms as a **2** of $SU(2)$. Therefore the states (B) transform under $SU(2)$ as $\mathbf{2} \times \mathbf{J} = (\mathbf{J} + \frac{\mathbf{1}}{\mathbf{2}}) \oplus (\mathbf{J} - \frac{\mathbf{1}}{\mathbf{2}})$. Furthermore, $a_1^\dagger a_2^\dagger$ is antisymmetric under interchange of $a_1^\dagger$ and $a_2^\dagger$ and so corresponds to the $\epsilon_{\alpha\beta} u^\alpha u^\beta$ **1** representation of $SU(2)$. Therefore the SU(2) content of the states is

$$a_1^\dagger a_2^\dagger |\Omega, J\rangle \longrightarrow J \qquad (2J+1)$$

$$a_1^\dagger |\Omega, J\rangle, a_2^\dagger |\Omega, J\rangle \longrightarrow (J + \frac{1}{2}) \oplus (J - \frac{1}{2}) \qquad (2J+2) + 2J$$

$$|\Omega, J\rangle \longrightarrow J \qquad (2J+1)$$

We therefore see that the massive susy reps have equal numbers of bosonic and fermionic states, as expected.

Let us consider some examples of massive representations of the supersymmetry algebra.

Massive Chiral Multiplet

We first suppose $|\Omega >= |P, 0\rangle$. As states we have

$$\text{Fermions: } a_1^\dagger |\Omega\rangle, a_2^\dagger |\Omega\rangle$$

$$\text{Bosons: } |\Omega\rangle, a_1^\dagger a_2^\dagger |\Omega\rangle$$

These give one massive scalar and one spin-(1/2) fermion. In a supersymmetric field theory, such a multiplet can be generated by giving a mass term to a chiral superfield, provided this is allowed by the gauge symmetries of the theory. Note that all states in this representation are in the same gauge representation. If this representation is real, then the multiplet can be CPT invariant, and in this case the particles have Majorana masses.

If the gauge representation is complex then CPT necessarily generates new particles, and so the complete massive multiplet involves four fermionic and four bosonic degrees of freedom, corresponding to a Dirac mass.

Vector Multiplet

In this case we start with $|\Omega\rangle = |P, \frac{1}{2}\rangle$. From our general analysis the matter content of the theory is now found to be

1× Massive spin-1 vector 31×Real scalar 12×Spin 1/2 fermion 4

As expected there are equal numbers (4) of bosonic and fermionic states.

Note that there are the same total number of states as for 1 massless vector multiplet and one massless chiral multiplet (why?).

4.6 Central Charges and BPS States

An interesting and important set of states occurs in the case that the supersymmetry algebra has *central charges*,

$$\{Q^L_\alpha, Q^M_\beta\} = \epsilon_{\alpha\beta} Z^{LM}. \tag{105}$$

Here Z is antisymmetric, $Z^{LM} = -Z^{ML}$. The antisymmetry of Z implies that this can only occur for $N \geq 2$ supersymmetry. By unitary rotations we can put Z in a standard form (N even)

$$Z = \begin{pmatrix} 0 & q_1 & 0 & 0 & \dots & \dots & 0 & 0 \\ -q_1 & 0 & 0 & 0 & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & q_2 & \dots & \dots & 0 & 0 \\ 0 & 0 & q_2 & 0 & \dots & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & \dots & 0 & q_{N/2} \\ 0 & 0 & 0 & 0 & \dots & \dots & -q_{N/2} & 0 \end{pmatrix}. \tag{106}$$

We will study this algebra for the top 2×2 block; the extension to the general case is then easy. We also will use Q for q_1. The supersymmetry algebra gives us

$$\{Q^1_1, Q^2_2\} = Z, \qquad \{Q^2_1, Q^1_2\} = -Z, \qquad \{Q^1_{\dot 1}, Q^2_{\dot 2}\} = Z, \qquad \{Q^2_{\dot 1}, Q^1_{\dot 2}\} = -Z.$$

$$\{Q^L_1, Q^M_{\dot 1}\} = \{Q^L_2, Q^M_{\dot 2}\} = 2M\delta^{LM}$$

We can rewrite this algebra by defining

$$a^1_\alpha = \frac{1}{\sqrt{2}} \left(Q^1_\alpha + \epsilon_{\alpha\dot\beta} \bar{Q}^2_{\dot\beta} \right).$$

That is:

$$a_1 = \frac{1}{\sqrt{2}} \left(Q^1_1 + \tilde{Q}^2_{\dot 2} \right)$$

$$a_2 = \frac{1}{\sqrt{2}} \left(Q^1_2 - \tilde{Q}^2_{\dot 1} \right)$$

Likewise we can define

$$b_1 = \frac{1}{\sqrt{2}} \left(Q^1_1 - \bar{Q}^2_{\dot 2} \right)$$

$$b_2 = \frac{1}{\sqrt{2}} \left(Q^1_2 + \bar{Q}^2_{\dot 1} \right)$$

and so

$$a^\dagger_1 = \frac{1}{\sqrt{2}} \left(\bar{Q}^1_{\dot 1} + Q^2_2 \right), \qquad a^\dagger_2 = \frac{1}{\sqrt{2}} \left(\bar{Q}^1_{\dot 2} - Q^2_1 \right).$$

$$b_1^\dagger = \frac{1}{\sqrt{2}}\left(\bar{Q}_{\dot{1}^1} - Q_2^2\right), \qquad b_2^\dagger = \frac{1}{\sqrt{2}}\left(\bar{Q}_{\dot{2}^1} + Q_1^2\right).$$

These ladder operators satisfy the relations

$$\{a_1, a_1\} = \{a_2, a_2\} = \{b_1, b_1\} = \{b_2, b_2\} = \{a_1, a_{\dot{2}}\} = \{b_1, b_{\dot{2}}\} = 0.$$

However

$$\begin{aligned} \{a_1, a_{\dot{1}}^\dagger\} &= \frac{1}{2}\{Q_1 + \bar{Q}_{\dot{2}}^2\} \\ &= \frac{1}{2}(2M + 2M + Z + Z) = 2M + Z. \end{aligned} \tag{107}$$

and we also find

$$\{a_2, a_{\dot{2}}^\dagger\} = 2M + Z. \tag{108}$$

Similar calculations give

$$\{b_1, b_{\dot{1}}^\dagger\} = -Z + 2M. \tag{109}$$

and in general we have

$$\begin{aligned} \{a_\alpha, a_{\dot{\beta}}^\dagger\} &= \delta_{\alpha\dot{\beta}}(2M + Z), \\ \{b_\alpha, b_{\dot{\beta}}^\dagger\} &= \delta_{\alpha\dot{\beta}}(2M - Z). \end{aligned} \tag{110}$$

This tells us that $2M \geq Z$, as otherwise we could not have a positive definite norm. To see this, suppose that $2M < Z$. Then for a state $|\phi\rangle$,

$$\langle\phi|b_1 b_{\dot{1}}^\dagger + b_{\dot{1}}^\dagger b_1|\phi\rangle < 0$$

and so

$$||b_{\dot{1}}^\dagger|\phi\rangle||^2 + ||b_1|\phi\rangle||^2 < 0,$$

which implies that there is at least one state with negative norm.

Furthermore, as there can be no physical zero norm states, then if $2M = Z$ then b and $b^\dagger$ must annihilate the state, and are represented by the zero operator.

A state having $2M = Z$ is called BPS, and for such a state only the $(a, a^\dagger)$ operators are active. In this case there are only four states in the representation in total $(|\Omega\rangle, a_1^\dagger|\Omega\rangle, a_2^\dagger|\Omega\rangle, a_1^\dagger a_2^\dagger|\Omega\rangle)$.

For general values of N, the algebra (110) becomes

$$\begin{aligned} \{a_\alpha^M, a_{\dot{\beta}}^{\dagger,L}\} &= \delta_{\alpha\dot{\beta}}\delta^{ML}(2M + Z_L)\,(\text{no summation over L}), \\ \{b_\alpha^M, b_{\dot{\beta}}^{\dagger,L}\} &= \delta_{\alpha\dot{\beta}}\delta^{ML}(2M - Z_L)\,(\text{no summation over L}). \end{aligned} \tag{111}$$

Here $M, L = 1 \ldots N/2$, where N is the total number of supersymmetry generators (e.g. $N = 8$ for maximal $N = 8$ supersymmetry). We can then reprise the above arguments to see that

1. $2M \geq Z_i$ for all i.
2. A general state with $2M > Z_i$ for all i has 2^{2N} states in the representation coming from use of the $2N$ pairs of creation/annihilation operators. Such a state is called a *long multiplet*.

3. A state with $2M = Z_i$ is called BPS. If there are r central charges with value Z_i, then there are $2N - r$ pairs of creation/annihilation operators and there are 2^{2N-r} states in the representation. Such a state is called a *short multiplet*.

4. A state for which $2M = Z_i$ for all i (which clearly requires $Z_i = Z_j$ for all i, j) is called an *ultrashort multiplet*. Such a multiplet has a total of 2^N states, and is the smallest possible susy multiplet.

BPS multiplets have two very useful properties. They are smaller than regular representations of the susy algebra and have mass/charge ratios that are fixed by the supersymmetry algebra. These properties are particularly useful for studying the strong coupling limit of supersymmetric theories, as these properties are protected under a continuous variation in the gauge coupling. This allows results to be deduced at strong coupling, where we might not otherwise be able to calculate.

5 Supersymmetric Field Theories

We now want to develop the formalism to construct and write supersymmetric field theories. To do so it is convenient (although not necessary) to develop the idea of *superspace*.

We start by introducing fermionic Grassmann coordinates θ^α. These will be viewed as fermionic coordinates on superspace. The bosonic coordinates are the regular spatial coordinates x^μ. We can write the supersymmetry algebra as

$$\left[\xi^\alpha Q_\alpha, \bar{\xi}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}\right] = 2\left(\xi_\alpha \sigma^m_{\alpha\dot{\alpha}} \bar{\xi}_{\dot{\alpha}}\right) P_m. \tag{112}$$

i.e.

$$\left[\xi Q, \bar{\xi}\bar{Q}\right] = 2(\xi\sigma^m\bar{\xi})P_m. \tag{113}$$

Note that these expressions involve commutators rather than anti-commutators. We can use this algebra to represent the supersymmetry algebra as translations on superspace.

Superspace is parametrised by

$$(X^m, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}).$$

Let us define a group element ('translations in superspace') by

$$G(x, \theta, \bar{\theta}) = \exp\left[i\left(-\underbrace{x^m P_m}_{spatial} + \underbrace{\theta Q + \bar{\theta}\bar{Q}}_{fermionic}\right)\right].$$

This represents superspace translations in both spatial and Grassmann directions.

What happens if we have two such transformations in superspace? Consider

$$G(0, \xi\bar{\xi})G(x^M, \theta, \bar{\theta}) = e^{i[\xi Q + \bar{\xi}\bar{Q}]} e^{i[-x^m P_m + \theta Q + \bar{\theta}\bar{Q}]}. \tag{114}$$

The Hausdorff formula is

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\text{higher commutators}} \tag{115}$$

As $[Q,P]=0$ and $[Q,\bar{Q}]=P$, all higher commutators vanish, and so it is straightforward to multiply the two elements, obtaining

$$G(0,\xi,\bar{\xi})G(x^M,\theta,\bar{\theta}) = G(x^M + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta}, \theta+\xi, \bar{\theta}+\bar{\xi}). \tag{116}$$

This therefore induces a motion in parameter space

$$g(\xi,\bar{\xi})(x^m,\theta,\bar{\theta}) \to (x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta}, \theta+\xi, \bar{\theta}+\bar{\xi}).$$

We can write this motion as

$$\begin{aligned} g(\xi,\bar{\xi}) = \xi Q + \bar{\xi}\bar{Q} &= \xi^\alpha \left(\frac{\partial}{\partial\theta_\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m\right) \\ &\quad + \bar{\xi}_{\dot{\alpha}}\left(\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma_{\alpha\dot{\beta}^m}\epsilon^{\dot{\beta}\alpha}\partial_m\right) \end{aligned} \tag{117}$$

These provide a representation of the supersymmetry algebra acting on superspace:

$$Q_\alpha = \frac{\partial}{\partial\theta_\alpha} - i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m, \tag{118}$$

$$\bar{Q}^{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}} - i\theta^\alpha\sigma^m_{\alpha\dot{\beta}}\epsilon^{\dot{\beta}\alpha}\partial_m, \tag{119}$$

with $\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2i\sigma_{\alpha\dot{\alpha}}$ (recall $P_m = i\partial_m$).

We can now define *superfields.* Superfields are essentially functions on superspace, understood as power series expansions in θ and $\bar{\theta}$.

$$\begin{aligned} \Phi(x,\theta,\bar{\theta}) &= \phi(x) + \theta\lambda(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta A(x) + \bar{\theta}\bar{\theta}B(x) \\ &\quad + \theta\sigma^m\bar{\theta}c_m(x) + \theta\theta\bar{\theta}\bar{\psi}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}D(x). \end{aligned} \tag{120}$$

Superfields transform under supersymmetry transformations in the natural way:

$$\delta_\xi\Phi(x,\theta,\bar{\theta}) \equiv \left(\xi Q + \bar{\xi}\bar{Q}\right) F.$$

Different types of superfields are defined by imposing different restrictions on the general superfield. To define these restrictions, it is very helpful to introduce the superspace derivatives D_α and $\bar{D}_{\dot{\alpha}}$,

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m \tag{121}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m. \tag{122}$$

As an exercise, we can check that D_α and $\bar{D}$ satisfy the anticommutation relations,

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^m_{\alpha\dot{\alpha}}\partial_m$$

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot{\alpha}}, \bar{D}_{\dot{\beta}}\} = 0.$$

Furthermore, D and Q anticommute:

$$\{D, Q\} = \{D, \bar{Q}\} = \{\bar{D}, Q\} = \{\bar{D}, \bar{Q}\} = 0.$$

We can now define the single most important form of superfield, the *chiral superfield*.

A chiral superfield is defined as a superfield whose components are restricted by

$$\bar{D}\Phi = 0. \tag{123}$$

Chiral superfields are the generalisation to field theory of the chiral multiplet.

The second most important kind of superfield is the vector superfield. This is defined by

$$V = V^\dagger.$$

Note that as $\bar{D}$ commutes with Q, the notion of a chiral superfield is preserved under supersymmetry transformations. As $(\xi Q + \bar{\xi}\bar{Q})^\dagger = (\xi Q + \bar{\xi}\bar{Q})$, the same is also true of vector superfields: the nature of a superfield is preserved under supersymmetry transformations.

Just as fields (scalars, spinors, vectors...) are the basic ingredients of ordinary quantum field theory, so superfields are the basic ingredient of supersymmetric quantum field theory. Superfields contain within themselves several different fields, all related by supersymmetry transformations.

5.1 Chiral Superfields

Recall chiral superfields are defined by

$$\bar{D}_{\dot{\alpha}}\Phi = 0.$$

As $\bar{D}$ is a linear differential operator, we immediately see that if Φ_1 and Φ_2 are chiral superfields, then

1. $\Phi_1 + \Phi_2$ is a chirla superfield.
2. $\Phi_1\Phi_2$ are chiral superfields.

Let us now look explicitly at the form of $\Phi(x)$. To solve $\bar{D}_{\dot{\alpha}}\Phi = 0$, it is convenient to write $y^m = x^m + i\theta\sigma^m\bar{\theta}$. This allows us to re-express (exercise)

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m \quad \rightarrow \quad \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial y^m}, \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m \quad \rightarrow \quad -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}. \end{aligned} \tag{124}$$

These expressions seem a bit confusing (in fact contradictory) at first, and so we should say a little about what they mean. They should be understood first as acting on a superfield $\Phi(x^m, \theta, \bar{\theta})$, and then transformed as acting on a superfield written as $\Phi(y^m, \theta, \bar{\theta})$. For example, the transformed version of $\bar{D}_{\dot{\alpha}}$ should be interpreted as the statement that $\bar{D}_{\dot{\alpha}}\theta = \bar{D}_{\dot{\alpha}}y = 0$. The significance of this is that it implies that the general chiral superfield can be expanded as a power series in y and θ,

$$\Phi(y,\theta,\bar{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{125}$$

If we further expand $y^m = x^m + i\theta\sigma^m\bar{\theta}$, we then have

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) &= \phi(x) + i\theta\sigma^m\bar{\theta}\partial_m\phi(x) + \frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\partial_\mu\partial^\mu\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) - \frac{i}{\sqrt{2}}\theta\theta\partial_m\psi(x)\sigma^m\bar{\theta} + \theta\theta F(x).\end{aligned} \tag{126}$$

As an exercise, check this expansion.

Let us count degrees of freedom, writing Φ in the form

$$\Phi = \underbrace{\phi(y)}_{\text{complex scalar}} + \underbrace{\sqrt{2}\theta\psi(y)}_{\text{fermions}} + \underbrace{\theta\theta F(y)}_{\text{auxiliary field}} \tag{127}$$

Off-shell, the scalar ϕ has 2 real degrees of freedom, the fermion ψ 4 real degrees of freedom, and the auxiliary scalar field F 2 real degrees of freedom. On-shell, these will reduce to 2 propagating bosonic degrees of freedom and 2 propagating fermionic degrees of freedom. The equations of motion eliminate two of the four fermionic degrees of freedom, and it turns out that the auxiliary field F is non-propagating can be integrated out. The vev of F is however of great interest as it is a measure of the magnitude of supersymmetry breaking that is present.

5.2 Constructing supersymmetric Lagrangians

We now want to construct supersymmetric Lagrangians. To do so, let us look at how superfields transform under supersymmetry transformations. Let's first consider a general superfield,

$$\begin{aligned}S(x,\theta,\bar{\theta}) &= \phi(x) + \theta\psi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta M(x) + \bar{\theta}\bar{\theta}N(x) \\ &\quad + \theta\sigma^\mu\bar{\theta}V_\mu(x) + \theta\theta\bar{\theta}\bar{\lambda}(x) + \bar{\theta}\bar{\theta}\theta\rho(x) + \theta\theta\bar{\theta}\bar{\theta}D(x).\end{aligned} \tag{128}$$

Under a susy transformation $\delta S = (\epsilon Q + \bar{\epsilon}\bar{Q})S$, giving

$$\begin{aligned}
\delta\phi &= \epsilon\psi + \bar{\epsilon}\bar{\chi}, \\
\delta\psi &= 2\epsilon M + (\sigma^\mu\bar{\epsilon})\left(i\partial_\mu\phi + V_\mu\right), \\
\delta\bar{\chi} &= 2\bar{\epsilon}N - (\epsilon\sigma^\mu)(i\partial_\mu\phi - V_\mu), \\
\delta M &= \bar{\epsilon}\bar{\lambda} - \frac{i}{2}\partial_\mu\psi\sigma^\mu\bar{\epsilon}, \\
\delta N &= \epsilon\rho + \frac{i}{2}\epsilon\sigma^\mu\partial_\mu\bar{\chi}, \\
\delta\bar{\lambda} &= 2\bar{\epsilon}D + \frac{i}{2}\left(\bar{\sigma}^\nu\sigma^\mu\bar{\epsilon}\right)\partial_\mu V_\nu + i(\bar{\sigma}^\mu\epsilon)\partial_\mu M, \\
\delta V_\mu &= \epsilon\sigma_\mu\bar{\lambda} + \rho\sigma_\mu\bar{\epsilon} + \frac{i}{2}\left(\partial^\nu\psi\sigma_\mu\bar{\sigma}_\nu\epsilon - \bar{\epsilon}\bar{\sigma}_\nu\sigma_\mu\partial^\nu\bar{\chi}\right). \\
\delta\rho &= 2\epsilon D - \frac{i}{2}\left(\sigma^\nu\bar{\sigma}^\mu\epsilon\right)\partial_\mu V_\nu + i\left(\sigma^\mu\bar{\epsilon}\right)\partial_\mu N, \\
\delta D &= \frac{i}{2}\partial_\mu\left(\epsilon\sigma^\mu\bar{\lambda} - \rho\sigma^\mu\bar{\epsilon}\right).
\end{aligned} \tag{129}$$

These *should be verified by the student.* Although the above expressions are a mild chore to derive, once done we can use these expressions to read off the transformations for any of the constrained superfields, such as chiral superfields, vector superfields, etc.

One very important feature of the above transformations is that

$$\delta D = \partial_\mu (\qquad). \tag{130}$$

That is, the auxiliary component D always transforms as a total derivative. The significance of this is that it implies that for any superfield $S(x, \theta, \bar{\theta})$,

$$\int d^4x d^2\theta d^2\bar{\theta} S(x, \theta, \bar{\theta})$$

is *invariant* under supersymmetry transformations.

We now have a procedure to construct (part of) supersymmetric Lagrangians.

1. Take any superfield
2. Take its D-term, and integrate over spacetime.

The resulting theory will automatically be invariant under supersymmetry transformations.

Now let's extend this a bit further by restricting the general susy transformations just to chiral superfields. In this case the susy transformations are

$$\begin{aligned}
\delta\phi &= \sqrt{2}\epsilon\psi, \\
\delta\psi &= i\sqrt{2}\sigma^\mu\bar{\epsilon}\partial_\mu\phi + \sqrt{2}\epsilon F, \\
\delta F &= i\sqrt{2}\bar{\epsilon}\bar{\sigma}^\mu\partial_\mu\psi.
\end{aligned} \tag{131}$$

Note that now for chiral superfields the F-term transforms as a total derivative, and also recall that products of chiral superfields are also chiral superfields. Consequently any holomorphic function $W(\Phi)$ of chiral superfields is also a chiral superfield, and

$$\int d^4x d^2\theta \, W(\Phi)$$

is invariant under supersymmetry transformations. Such a term is called an *F-term*, whereas terms of the form

$$\int d^4x d^2\theta d^2\bar{\theta} \, K(\Phi, \bar{\Phi})$$

are called *D-terms*.

Suppose we have a set of n chiral superfields $\Phi_1, \ldots Phi_n$. We can construct supersymmetric Lagrangians using two functions

1. $K(\Phi_i, \Phi_j^\dagger)$ - a real, non-holomorphic function of superfields and their conjugates
2. $W(\Phi_i)$ - a complex holomorphic function of chiral superfields.

Then a supersymmetric Lagrangian is given by

$$\mathcal{L} = \underbrace{\int d^4x d^2\theta d^2\bar{\theta}\, K(\Phi_i, \Phi_j^\dagger)}_{D-term} + \underbrace{\int d^4x d^2\theta\, W(\Phi)}_{F-term}. \tag{132}$$

This is in fact the most general supersymmetric Lagrangian for a theory of chiral superfields. What are the mass dimensions of fields?

$$\Phi(y, \theta, \bar{\theta}) = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{133}$$

Here ϕ has mass dimension 1 as a scalar, and ψ as a fermion has mass dimension 3/2. Consequently θ has mass dimension $-1/2$, while F has mass dimension 2. We also see that K has mass dimension 2 and W mass dimension 3.

A renormalisable theory therefore has

$$K = \sum c_{ij}\Phi^i\Phi^{j,\dagger}, \tag{134}$$

$$W = \sum_i \lambda_i \Phi^i + \underbrace{\sum m_{ij}\Phi^i\Phi^j}_{\text{mass terms}} + \underbrace{\sum Y_{ijk}\Phi^i\Phi^j\Phi^k}_{\text{Yukawa couplings}}. \tag{135}$$

We will shortly discuss some simple supersymmetric field theories. Before doing so, we first state one truly remarkable result that explains why supersymmetric theories are so attractive.

The superpotential is not renormalised at any order in perturbation theory.

It is tree-level exact up to non-perturbative corrections.

The Kähler potential by contrast gets renormalised at all orders in perturbation theory.

This point is so important it is worth dwelling on. First, what is the physics of this? The underlying reason for non-renormalisation is holomorphy. Perturbation theory is an expansion in a coupling constant, which is real, while the superpotential is holomorphic and complex. So a *real* expansion cannot be made consistent with holomorphy, and so the holomorphic quantity (the superpotential) therefore cannot be renormalised. In contrast, the Kähler potential (which is real) does indeed get renormalised at every order in perturbation theory.

Let us now make a slight diversion in order to understand better how this occurs. We will deal with the expansion in powers of gauge coupling.

We start by promoting gauge couplings to superfields. In string theory, this always happens - all coupling constants originate as the vevs of very weakly (gravitationally) coupled fields. In the context of globally supersymmetric field theory, such ultra-weakly coupled fields are often called *spurion* fields.

Now consider the action of a non-Abelian gauge theory,

$$\frac{1}{g^2}\int d^4x F^a_{\mu\nu}F^{a,\mu\nu} + i\theta_{YM}\int d^4x F^a_{\mu\nu}(*F)^{a,\mu\nu}. \tag{136}$$

Scalar superfields are naturally chiral superfields involving complex components. The gauge coupling by itself is real, and is naturally complexified by including the θ-angle. So a gauge coupling corresponds to a chiral superfield

$$\Phi = (\frac{1}{g^2} + i\theta_{YM}) + \dots$$

Now, we know from standard field theory arguments that the topological term $\int d^4 x F^a_{\mu\nu} (*F)^{a,\mu\nu}$ is locally a total derivative and vanished for field configurations that are continuously connected to the vacuum. So when we perform the path integral

$$\int \mathcal{D}A e^{-\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu} + i\theta F_{\mu\nu} (*F)^{\mu\nu}} \tag{137}$$

perturbative fluctuations about $A_\mu = 0$ do not depend on the value of θ. (Note that non-perturbative instantonic effects *do* depend on θ, and contribute to the action terms that are suppressed as $e^{-\frac{8\pi}{g^2} + i\theta}$) As a result, the perturbative expansion depends only on $g^{-2} = \Phi + \bar{\Phi}$ and *not at all* on θ.

Now comes the magic part. The superpotential is holomorphic, and so must be a holomorphic function of Φ. However, we have just established that perturbation theory is an expansion in $\Phi + \bar{\Phi}$, and so can never modify a holomorphic quantity.

We therefore conclude: the superpotential is not renormalised at any order in perturbation theory.

This is an *AMAZING* result!

5.3 The Wess-Zumino Model

We now want to be a little more prosaic and examine in some detail the simplest example of a supersymmetric field theory. We consider a theory of a single chiral superfield with

$$\begin{aligned} K &= \Phi^\dagger \Phi, \\ W &= m\Phi^2 + g\Phi^3. \end{aligned} \tag{138}$$

What is the explicit form of this? Recall that we can expand Φ as

$$\begin{aligned} \Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \\ &= \phi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x) + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\phi \\ &\quad -\frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\phi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta}. \end{aligned} \tag{139}$$

We can now look for the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component of $\Phi^\dagger\Phi$ (exercise!), and find

$$\Phi^\dagger\Phi_{\theta\theta\bar{\theta}\bar{\theta}} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi^*) - \frac{1}{4}\phi^* (\partial_\mu\partial^\mu\phi) - \frac{1}{4}\phi (\partial_\mu\partial^\mu\phi^*) + |F|^2 + \frac{i}{2}\left(\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + \partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi\right). \tag{140}$$

Integrating by parts, we obtain

$$\Phi^\dagger\Phi_{\theta\theta\bar{\theta}\bar{\theta}} = \left[(\partial_\mu\phi)(\partial^\mu\phi^\dagger) + |F|^2 - i(\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi)\right] \tag{141}$$

and so

$$\int d^4x d^2\theta d^2\bar{\theta}\Phi^\dagger\Phi = \int d^4x \partial_\mu\phi\partial^\mu\phi^\dagger + |F|^2 - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi. \tag{142}$$

We see that this gives rise to standard kinetic terms for the scalar and fermionic fields. These we know and love. There is also an unusual term that we have not seen before, involving $|F|^2$ but no kinetic terms for the F field.

Let us pause on understanding this, and instead consider the F-terms involving the superpotential. These take the basic form

$$\int d^4x\, d^2\theta W(\Phi) \tag{143}$$

We can again recall (again and again) the basic expansion

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y). \tag{144}$$

This time however, as $y = x + \theta\sigma^\mu\bar{\theta}$, in terms of the expansion it is sufficient to replace y by x, as none of the
$bar\theta$ terms can survive the superspace integral. We can then expand in $(\Phi - \phi)$, and write $\Phi = \phi(x) + (\Phi - \phi)(x)$.

$$W(\Phi) = W(\phi) + \underbrace{(\Phi - \phi)}_{\sqrt{2}\theta\psi + \theta\theta F}\frac{\partial W}{\partial\phi} + \underbrace{\frac{1}{2}(\Phi - \phi)^2}_{-\frac{1}{2}(\theta\theta)(\psi\psi)}\frac{\partial^2 W}{\partial\phi^2}. \tag{145}$$

Then

$$\int d^4x d^2\theta W(\Phi) + \text{c.c} = \left(\frac{\partial W}{\partial\phi}|_{\Phi=\phi}F + \text{c.c}\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{c.c}\right) \tag{146}$$

The F-term Lagrangian density is then

$$\mathcal{L} = \partial_\mu\phi\partial^\mu\phi^* - i\left(\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi\right) + |F|^2 + \left(\frac{\partial W}{\partial\phi}F + \frac{\partial W^*}{\partial\phi^*}F^*\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\phi^2}\psi\psi + \text{c.c}\right) \tag{147}$$

We can combine (147) with (142) to see that the auxiliary field F appears in the Lagrangian as

$$\mathcal{L}_F = FF^* + F\frac{\partial W}{\partial\phi} + F^*\left(\frac{\partial W}{\partial\phi}\right)^*. \tag{148}$$

As there are no derivative terms for F, we see that F is not a propagating degree of freedom. At least classically, we can simply solve for the equations of motion of F and integrate it out.[1]

The equations of motion give

$$0 = \frac{\partial\mathcal{L}}{\partial F} = F^* + \frac{\partial W}{\partial\phi}, \tag{149}$$

[1]Quantum mechnically, we will have to be a little more careful when performing the path integral. In particular, it is necessary to account correctly for the measure when integrating over $D\mathcal{F}$.

and so

$$F^* = -\frac{\partial W}{\partial \phi}. \tag{150}$$

The Lagrangian $\mathcal{L}_F$ therefore becomes

$$\mathcal{L}_F = -\left(\frac{\partial W}{\partial \phi}\right)\left(\frac{\partial W^*}{\partial \phi^*}\right). \tag{151}$$

We then obtain overall

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - i\bar{\psi}\bar{\sigma}^\mu \partial_\mu \psi - \left(\frac{\partial W}{\partial \phi}\right)\left(\frac{\partial W^*}{\partial \phi^*}\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial \phi^2}\psi\psi + \text{c.c}\right). \tag{152}$$

What does this tell us? There are kinetic terms for both scalar and fermion fields, and interaction terms between scalars and fermions. There is also however a potential, and the potential is given by

$$V_F = |F|^2 = \left|\frac{\partial W}{\partial \phi}\right|^2. \tag{153}$$

There are some important features about this potential that will generalise to all supersymmetric models which do not include gravity.[2]

1. The scalar potential is positive semi-definite, and cannot be negative.
2. The scalar potential is given by the (sum over) squares of the susy breaking parameters, here the F-term.
3. The scalar potential vanishes if and only if supersymmetry is unbroken.

Let us now be a little more concrete. We take the specific example of

$$W = \frac{m\Phi^2}{2} + \frac{g\Phi^3}{3}. \tag{154}$$

This is the superpotential appropriate to the Wess-Zumino model. Then the scalar potential is given by

$$V = \left|\frac{\partial W}{\partial \phi}\right|^2 = m^2 \Phi\Phi^* + gm\left(\Phi\Phi^{*,2} + \Phi^2\Phi^*\right) + g^2\Phi^2\Phi^{*,2}. \tag{155}$$

Note that this scalar potential has

1. A mass term $m^2\Phi\Phi^*$.
2. A quartic scalar interaction $g^2\Phi^2\Phi^{*,2}$.

Now consider the fermionic terms from the Lagrangian (152). We have

$$\frac{\partial^2 W}{\partial \phi^2} = m + 2g\Phi. \tag{156}$$

[2] A sceptical reader may observe here that this is like proving results that hold for all fluids with zero viscosity. The scepticism may have a point.

This then gives a fermionic interaction Lagrangian of

$$\mathcal{L}_{int} = -\frac{1}{2}\left(m\psi\psi + m\bar{\psi}\bar{\psi}\right) - g\left(\Phi\psi\psi + \bar{\Psi}\bar{\psi}\bar{\psi}\right). \tag{157}$$

This gives a

1. mass term $-\frac{1}{2}\left(m\psi\psi + m\bar{\psi}\bar{\psi}\right)$ and so the fermion mass is m.
2. Yukawa coupling $g(\Phi\psi\psi + \bar{\Phi}\bar{\psi}\bar{\psi})$.

In particular, notice - and absorb! - the fact that the coefficient of the trilinear Yukawa coupling is essentially the square-root of the coefficient of the quartic scalar interaction.

XX FIT IN FEYNMAN DIAGRAMS XX

This is at the heart of the famous supersymmetric cancellations in the radiative corrections to the Higgs mass.

Note that supersymmetry does *not* mean that any interaction can be modified simply by swapping bosons and fermions - it is instead far more sophisticated than that. Instead it relates the quartic scalar interaction to the trilinear Yukawa interaction.

Let us now examine a little more closely the general structure of supersymmetric theories. Recall that for a chiral multiplet

$$\Phi = \phi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y), \tag{158}$$

then the supersymmetry transformations act as

$$\begin{aligned} \delta\phi &= \sqrt{2}\epsilon\psi, \\ \delta\psi &= \sqrt{2}i\sigma^\mu\bar{\epsilon}\partial_\mu\phi + \sqrt{2}\epsilon F, \\ \delta F &= i\sqrt{2}\epsilon\sigma^\mu\partial_\mu\psi. \end{aligned} \tag{159}$$

Also recall that in general for a symmetry transformation $\mathcal{Q}$ to be unbroken , it must preserve the vacuum and so

$$\mathcal{Q}|0\rangle_{vac} = 0. \tag{160}$$

For a general field background, we see from (159) that we always require $\langle\psi\rangle = 0$ in order to preserve supersymmetry. This is no great hardship, as we require $\langle\psi\rangle = 0$ to preserve Lorentz symmetry. In a similar vein, the Lorentz condition $\partial_\mu\phi = 0$ also eliminates a further term. For Lorentz-preserving field configurations we are then left with

$$\begin{aligned} \delta\phi &= 0, \\ \delta\psi &= \sqrt{2}\epsilon F, \\ \delta F &= 0. \end{aligned} \tag{161}$$

From this it is straightforward to see that $\langle F\rangle$ is an order parameter of supersymmetry breaking.

For a canonical Kähler potential it is straightforward to generalise the above analysis for a single superfield to the case of n superfields. In this case the auxiliary field Lagrangian is

$$\mathcal{L}_F = \sum_i F_i F_i^\dagger + \sum_i \left(F_i \frac{\partial W}{\partial \phi_i} + F_i^\dagger \frac{\partial W}{\partial \phi_i^\dagger} \right). \tag{162}$$

From these we obtain that

$$F_i^\dagger = -\frac{\partial W}{\partial \phi}, \quad F_i = -\frac{\partial W}{\partial_i^\dagger},$$

and so

$$V = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2. \tag{163}$$

For general forms of the Kähler potential, this instead becomes

$$V = \sum K^{i\bar{j}} \left(\frac{\partial W}{\partial \phi_i} \right) \left(\frac{\partial W}{\partial \phi^*_{\bar{j}}} \right). \tag{164}$$

Here the Kähler metric is defined by

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial \phi_i \partial \phi_{\bar{j}}}. \tag{165}$$

General properties of (globally) supersymmetric theories are

1. Susy breaking is parametrised by the F-terms $F_i = -\frac{\partial W}{\partial \phi_i}$. Supersymmetry is unbroken if and only if $\langle F_i \rangle = 0$ for all i.
2. The potential is positive semi-definite,
$$V = \sum K^{i\bar{j}} \left(\frac{\partial W}{\partial \phi_i} \right) \left(\frac{\partial W}{\partial \phi^*_{\bar{j}}} \right). \tag{166}$$
3. The potential vanishes *if and only if* we have a supersymmetric solution.

One very natural question to ask about supersymmetric theories is whether supersymmetric solutions always exist. This is not a question to deliberate on, as the answer is straightforward: No. There is a simple and well-known counterexample, the O'Raifertaigh model. This is defined by

$$K = \sum_{i=1}^{3} \Phi_i^\dagger \Phi_i, \tag{167}$$

$$W = g\Phi_1 \left(\Phi_3^2 - m^2 \right) + M\Phi_2\Phi_3. \tag{168}$$

The F-terms are then given by

$$-F_1^* = \tfrac{\partial W}{\partial \Phi_1}|_{\Phi=\phi} = g(\phi_3^2 - m^2), \tag{169}$$

$$-F_2^* = \tfrac{\partial W}{\partial \Phi_2}|_{\Phi=\phi} = M\phi_3, \tag{170}$$

$$-F_3^* = \tfrac{\partial W}{\partial \Phi_3}|_{\Phi=\phi} = 2g\phi_1\phi_3 + M\phi_2. \tag{171}$$

It is easy to see that this theory *cannot* preserve supersymmetry as we cannot solve $\langle F_i \rangle = 0$ simultaneously.

As an exercise, study the potential and spectrum of this model. For simplicity take $m^2 < \frac{M^2}{2g^2}$. The minimum is found to be at $\langle phi_2 \rangle = \langle \phi_3 \rangle = 0$, with $\langle \phi_1 \rangle$ unfixed.

The fermionic spectrum is given by

$$(0, M, M),$$

while the bosonic spectrum is given by

$$(0, \sqrt{M^2 + 2gm^2}, \sqrt{M^2 - 2gm^2}).$$

The magnitude of the scalar potential is

$$V_{min} = |F_1|^2 = g^2 m^4. \tag{172}$$

The spectrum is non-degenerate, with the lack of degeneracy measured by $gm^2 = |F_1|$. Note that the form the non-degeneracy takes is to split the scalar masses by equal amounts above and below those of the fermions.

This is an example of a more general result, which is a supersymmetric sum rule that holds for such renormalisable, globally supersymmetric field theories.

$$STr(M^2) = \sum_b m^2 - \sum_f m^2 = 0. \tag{173}$$

That is, the sum over squared masses with a relative sign between bosonic and fermionic degrees of freedom always vanishes.

A second, and more trivial, example of supersymmetry breaking is the Fayet model. This is given by

$$\begin{aligned} K &= \Phi^\dagger \Phi, & (174) \\ W &= \lambda \Phi. & (175) \end{aligned}$$

This has an F-term $F_\phi = \lambda$ with $V = |F|^2 = \lambda 62 = V_0$. This has a trivial (flat) potential, with the vacuum energy simply shifted up. It has a massless scalar and a massless fermion, and no non-trivial interactions.

5.4 General Properties of Supersymmetric Theories

We are familiar with the way that Goldstone bosons arise from the spontaneous breaking of symmetries. Goldstone's theorem tells us that when a global symmetry is spontaneously broken, there remains a massless boson (a Goldstone boson) and that the masslessness of this boson is a direct consequence of the symmetry breaking.

Mutatis mutandis, the same is true of supersymmetry. The chief difference (pretty much the only difference really) is that supersymmetry is a fermionic symmetry rather than a bosonic symmetry. The parameter of supersymmetry transformations, ϵ, transforms as a spinor under the Lorentz group. The consequence of this is that any supersymmetric field theory with broken supersymmetry always has a massless fermion in its spectrum, a *goldstino*.

If you are awake reading this, this should sound bad. Massless particles can be probed both cosmologically and in collider experiments, and if such a massless fermion existed then we would expect to know about it. There is however a catch. In any supersymmetric theory with gravity - in other words, any supersymmetric theory with a chance to describe the real world - there is a *gravitino*. The gravitino is the supersymmetric partner of the graviton. The massless gravitino multiplet only has two degrees of freedom, but in theories with broken supersymmetry the gravitino becomes massive which required four degrees of freedom. It obtains the extra two degrees of freedom by eating the goldstino, in a supersymmetric analogue of the Higgs mechanism imaginatively named the super-Higgs mechanism.

We don't have time here for a full discussion of supergravity, but let us briefly mention how ideas from global supersymmetry extend to local supersymmetry. We will give results rather than derivations. First, chiral superfields remain chiral superfields.

$$\Phi \longrightarrow \Phi$$

The Kähler and superpotentials are in general non-renormalisable, so we should write

$$K = K(\Phi_i, \Phi_j^\dagger) \rightarrow M_P^2 K\left(\frac{\Phi_i}{M_P}, \frac{\Phi_j^\dagger}{M_P}\right), \qquad (176)$$

$$W = W(\Phi) \rightarrow M_P^3 W\left(\frac{\Phi}{M_P}\right). \qquad (177)$$

The F-terms again are the order parameters for supersymmetry breaking. However these are now given by

$$F_\Phi = \left(\partial_\Phi W + \frac{(\partial_\Phi K)}{M_P^2} W\right) e^{K/2M_P^2}. \qquad (178)$$

It is customary to write this as a covariant derivative, and also to note that F-terms can be raised and lowered using the Kähler metric,

$$F^i = e^{K/2} K^{i\bar{j}} D_{\bar{j}} W \equiv e^{K/2M_P^2} K^{i\bar{j}} \left(\partial_{\bar{j}} W + \left(\partial_{\bar{j}} K\right) W\right). \qquad (179)$$

As before supersymmetry is broken if $F^\Phi \neq 0$. However, one big (and phenomenologically important) difference is that the vacuum energy is no longer positive semi-definite. Instead, it can be written as

$$V = \underbrace{\sum_{i,j} K_{i\bar{j}} F^i F^{\bar{j}}}_{\sum |F|^2} - \frac{3}{M_P^2} \underbrace{e^{K/M_P^2} |W|^2}_{\text{supergravity term}} \qquad (180)$$

The last term is specific to supergravity. There are two crucial consequences of the negativity of the last term. First, supersymmetric solutions ($F^I = 0$ for all I) can have *negative* vacuum energy in supergravity. Secondly, solutions with broken supersymmetry can have vanishing vacuum energy. Given that our universe has vanishing vacuum energy, this implies that supersymmetry is not inconsistent with the world as we observe it.

The gravitino mass in supergravity is given by

$$m_{3/2} = e^{K/2M_P^2} \frac{|W|}{M_P^2}. \tag{181}$$

If the vacuum energy vanishes, then the gravitino mass is the order parameter of supersymmetry breaking. However also note that in supersymmetric theories with negative cosmological constant, the gravitino mass can be non-zero even while the graviton remains massless. At first this seems a little paradoxical, but it is in fact a consequence of the properties of AdS space (i.e. that with a negative cosmological constant). In AdS space supersymmetry is consistent with - and in fact requires - mass splittings between superpartners.

In theories with vanishing vacuum energy and broken supersymmetry, the gravitino mass is the order parameter of supersymmetry breaking and measures the magnitude of supersymmetry breaking. This is easy to see from the potential (180): if the vacuum energy vanishes, then

$$3m_{3/2}^2 M_P^2 = |F|^2, \tag{182}$$

and so the gravitino mass is a direct measure of the magnitude of the F-terms. As mentioned previously, the gravitino obtains its mass by eating the massless goldstino in the super-Higgs effect. The goldstino provides the two extra degrees of freedom necessary to convert a massless gravitino (two propagating degrees of freedom) into a massive gravitino (four propagating degrees of freedom).

Let us now describe one of the simplest non-trivial but important supergravity theories. This is the simplest possible example of a no-scale model. It is specified by (we put $M_P = 1$)

$$\begin{aligned} K &= -3\ln(T+\bar{T}), & (183)\\ W &= W_0. & (184) \end{aligned}$$

Here W_0 is a constant. There is one superfield T, and note that the theory only depends on the real part of T: there is a symmetry $T \to T + i\lambda$ in the theory. Then

$$K_{,T} = \frac{-3}{T+\bar{T}} \implies K_{,T\bar{T}} = \frac{3}{(T+\bar{T})^2}. \tag{185}$$

As a consequence we evaluate the F-term to get

$$\begin{aligned} F_T &= e^{K/2}\left(\partial_T W + \partial_T K W\right) & (186)\\ &= \frac{1}{(T+\bar{T})^{3/2}} \times \frac{-3W_0}{T+\bar{T}} = -\frac{3W_0}{(T+\bar{T})^{5/2}}. & (187) \end{aligned}$$

By raising indices using the Kähler metric, we have $F^T = -\frac{W_0}{(T+\bar{T})^{1/2}}$. It then follows that

$$K_{T\bar{T}} F^T F^{\bar{T}} = \frac{3W_0^2}{(T+\bar{T})^3}, \tag{188}$$

and as a consequence

$$V = K_{T\bar{T}} F^T F^{\bar{T}} - 3e^K W^2 \tag{189}$$

$$= \frac{3W_0^2}{(T+\bar{T})^3} - \frac{3W_0^2}{(T+\bar{T})^3} = 0. \tag{190}$$

The potential therefore vanishes for all values of the field T. The gravitino mass is given by

$$m_{3/2} = e^{K/2} W = \frac{W_0}{(T+\bar{T})^{3/2}}, \tag{191}$$

and is non-zero for all values of T.

This model is a no-scale model. There are several key features that define it as a no-scale model. The three most important features of a no-scale model are

1. Vanishing cosmological constant - as we see above, the solution has zero vacuum energy.
2. Broken supersymmetry - $F^T \neq 0$ and so the vacuum does not preserve supersymmetry. Equivalently, (given the vanishing vacuum energy) the gravitino mass is non-zero.
3. An unfixed flat direction. No scale models always have a residual unfixed direction which, at least at tree level, remains massless. In this case this is given by the field Re(T). Note that Im(T) is also massless, but this is guaranteed by the symmetry $T \to T + i\lambda$.

Originally there was a hope that no-scale models could provide a possible solution to the cosmological constant problem, as they combine a vanishing vacuum energy with broken supersymmetry. However it was soon realised that this is only a property of the tree-level theory, and at loop level the cosmological constant problem will return. At loop level the unfixed modulus T will also obtain a potential from corrections to the Kähler potential, thereby lifting the flatness of the potential.

Another reason for their importantce is that no scale models frequently occur in string theory, in the context of dimensional reduction of ten dimensional string theory solutions to 4 dimensions. So while the properties of no-scale solutions are non-generic if we consider the list of all possible supergravity theories, in the context of string theories they in fact occur regularly. In this context the shift symmetry $T \to T + \imath\lambda$ is perturbatively exact, and is equivalent to the symmetry in the QCD theta angle.

5.5 Vector Superfields

So far we have constructed chiral superfields and used these to build up Lagrangains describing supersymmetric theories of bosons and fermions. We now want to generealise this to theories involving gauge interactions and charges particles. To do so we need the concept of a vector multiplet.

The vector multiplet is defined by constraining the general multiplet by

$$V = V^\dagger.$$

This constrains the fields present in a general multiplet to be

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + i(\theta\chi(x) - \bar\theta\bar\chi(x)) \\ &+\frac{i}{2}\left[\theta\theta(M(x)+iN(x)) - \bar\theta\bar\theta(M(x)-iN(x))\right] \\ &- \left(\theta\sigma^m\bar\theta\right)V_m(x) + i\theta\theta\bar\theta\left[\lambda(x) + \frac{i}{2}\bar\sigma^m\partial_m\chi(x)\right] \\ &-i\bar\theta\bar\theta\theta\left[\lambda(x) + \frac{i}{2}\sigma^m\partial_m c\bar{h}i(x)\right] + \frac{1}{2}\theta\theta\bar\theta\bar\theta(D(x) + \frac{1}{2}\Box C(x)). \end{aligned} \tag{192}$$

Note the particular choice of fields that are used when writing the vector multiplet - for example, we have written $D(x)+\frac{1}{2}\Box C(x)$ instead of $D(x)$. This choice is motivated in advance by the fact that the fields $C(x), \chi(x), M(x), N(x)$ are all unphysical and can be gauged away.

Let us make this explicit. It is manifest that given a chiral superfield $\Phi(x)$, $\Phi(x) + \Phi^\dagger(x)$ is a vector superfield. In fact,

$$\begin{aligned} \Phi + \Phi^\dagger &= (A + A^\dagger) + \sqrt{2}\left(\theta\psi + \bar\theta\bar\psi\right) + \theta\theta F + \bar\theta\bar\theta F^* \\ &+i\theta\sigma^m\bar\theta\partial_m(A - A^*) + \frac{i}{\sqrt{2}}\theta\theta\bar\theta\bar\sigma^m\partial_m\psi \\ &+\frac{i}{\sqrt{2}}\bar\theta\bar\theta\theta\sigma^m\partial_m\bar\psi + \frac{1}{4}\theta\theta\bar\theta\bar\theta\Box\left(A + A^\dagger\right). \end{aligned} \tag{193}$$

The thing to note in particular here is that the $\theta\sigma^\mu\bar\theta$ part of $\Phi + \Phi^\dagger$ has the component $i\partial_m(A - A^\dagger)$. This looks precisely like a conventional gauge transformation, with $A(x)$ the scalar field that plays the role of the gauge parameter.

This motivates the supersymmetric generalisation of the notion of a gauge transformation:

Conventional:	$A_\mu \to A_\mu + \partial_\mu\phi$	for some ϕ,
Susy:	$V(x) \to V(x) + \Phi(x) + \Phi^\dagger(x)$	for some Φ.

Under this transformation, the elements of a vector superfield transform as

$$\begin{aligned} C(x) &\to C(x) + (A(x) + A^*(x)), \\ \chi(x) &\to \chi(x) - i\sqrt{2}\psi(x), \\ M + iN &\to (M + iN) - 2iF, \\ v_m &\to v_m - i\partial_m(A - A^*), \\ \lambda &\to \lambda, \\ D &\to D. \end{aligned} \tag{194}$$

From this we see that this generalises the traditional notion of a gauge transformation to supersymmetric theories. We also see that we can drastically simplify the form of a vector superfield by going to a particular gauge (where $C = \chi = M = N = 0$). This gauge is traditionally calles Wess-Zumino gauge.

In it,

$$\begin{aligned} V &= -\theta\sigma^m\bar{\theta}v_m(x) + i\theta\theta\bar{\theta}\bar{\lambda}(x) \\ &\quad -i\bar{\theta}\bar{\theta}\theta\lambda(x) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x), \qquad (195) \\ V^2 &= -\frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})v_\mu v^\mu, \qquad (196) \\ V^3 &= 0. \qquad (197) \end{aligned}$$

Note that the choice of gauge is *not* preserved under susy transformations. If you act on a vector field in Wess-Zumino gauge with a supersymmetry transformation parametrised by ϵ, the resulting field will no longer be in Wess-Zumino gauge.

We should view V as the superfield equivalent of the gauge potential, and $(\Phi + \Phi^*)$ as the supersymmetric analogue of a gauge transformation.

As with electrodynamics, we want to construct Lagrangians that are invariant under gauge transformations. For Abelian theories, the answer is reasonably easy to guess. Suppose a chiral multiplet A transforms under a gauge transformation as

$$A \to e^{iq\Lambda(x)}A.$$

This is the natural transformation law, that generalises the scalar transformation $\phi(x) \to e^{iq\lambda(x)\phi(x)}$. Recall that the kinetic term entering the Kähler potential is $A^\dagger A$. We then have the transformation

$$A^\dagger A(x) \to A^\dagger A(x) e^{iq(\Lambda - \Lambda^\dagger)}. \qquad (198)$$

We want to construct a gauge invariant kinetic term. Given that V transforms as $V \to V + i(\Lambda - \Lambda^\dagger)$, then under gauge transfomations

$$A^\dagger e^{-qV} A \to A^\dagger e^{-qV} A. \qquad (199)$$

This term then gives the gauge interactions between charged matter fields (A) and Abelian vector bosons V. Written out in components, we have

$$\begin{aligned} (\Phi^\dagger e^{-qV}\Phi)_{\theta\theta\bar{\theta}\bar{\theta}} &= FF^* + \phi\partial_\mu\partial^\mu\phi^* + i\partial_\mu\bar{\psi}\bar{\sigma}^\mu\psi \\ &\quad -qv^\mu\left(\frac{1}{2}\bar{\psi}\bar{\sigma}^\mu\psi + \frac{i}{2}\phi^*\partial_\mu\phi - \frac{i}{2}\left(\partial_\mu\phi^*\right)\phi\right) \qquad (200) \\ &\quad + \frac{i}{\sqrt{2}}q\left(\phi\bar{\lambda}\bar{\psi} - \phi^*\lambda\psi\right) + \frac{1}{2}\left(qD - \frac{1}{2}q^2 v_\mu v^\mu\right)\phi^*\phi. \end{aligned}$$

These are the appropriate interactions to couple a complex scalar and chiral fermion to gauge bosons and gauginos.

We also need the kinetic terms for the gauge field itself: that is, we need to find a supersymmetric multiplet that incorporates the $F_{\mu\nu}F^{\mu\nu}$ term representing the gauge kinetic term. The easiest way to do this is to find a superfield that includes the gauge field strength as one of its components. Consider the superfield

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}D_\alpha V. \qquad (201)$$

This is constructed from vector superfields and has the following properties. First, W_α is a chiral superfield. To see this, note that

$$\bar{D}_{\dot{\beta}} W_\alpha = 0, \tag{202}$$

as $\{\bar{D}_{\dot{\beta}}, \bar{D}_{\dot{\beta}}\} = 0$, and W_α by definition includes both $\bar{D}$ derivatives. Secondly, W_α is gauge invariant under the transformation

$$V \to V + \Phi + \Phi^\dagger.$$

To see this, note that W_α transforms as

$$W_\alpha \to -\frac{1}{4}\bar{D}\bar{D}D_\alpha \left(V + \Phi + \Phi^\dagger\right).$$

Now $D_\alpha \Phi^\dagger = 0$ as $\Phi^\dagger$ is an antichiral superfield. As a result we have

$$\begin{aligned} W_\alpha &\to -\frac{1}{4}\bar{D}\bar{D}D_\alpha V - \frac{1}{4}\bar{D}\bar{D}D_\alpha \Phi, \\ &= W_\alpha - \frac{1}{4}\bar{D}\left(\bar{D}D_\alpha + D_\alpha \bar{D}\right)\Phi, \qquad (\text{ as } \bar{D}\Phi = 0) \\ &= W_\alpha - \frac{1}{4}\bar{D}^{\dot{\beta}}\left(\bar{D}_{\dot{\beta}} D_\alpha + D_\alpha \bar{D}_{\dot{\beta}}\right)\Phi \\ &= W_\alpha - \frac{1}{4}\bar{D}^{\dot{\beta}}\{D_\alpha, \bar{D}_{\dot{\beta}}\}\Phi \\ &= W_\alpha - \frac{1}{4}\partial_\mu \left(\bar{D}^{\dot{\beta}}\Phi\right)\sigma^\mu_{\alpha\dot{\beta}} \\ &= W_\alpha. \qquad (\text{ as } \bar{D}^{\dot{\beta}}\Phi = 0). \end{aligned} \tag{203}$$

Therefore we establish that W_α is a *chiral*, *gauge-invariant* superfield.

Proving that W_α is gauge-invariant makes the evaluation of W_α a lot simpler. All we need to do is to go to Wess-Zumino gauge, in which the form of the vector superfield simplifies dramatically. Having done this, we can then evaluate W_α explicitly by taking the super-derivatives. If we do this, we find

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + D(y)\theta_\alpha + \theta\theta\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\bar{\lambda}^{\dot{\alpha}} \\ &\quad -\frac{i}{2}\left(\sigma^\mu\bar{\sigma}^\nu\right)_\alpha{}^\beta \underbrace{\left(\partial_\mu v_\nu(y) - \partial_\nu v_\mu(y)\right)}_{\text{Field strength } F_{\mu\nu}}\theta_\beta. \end{aligned} \tag{204}$$

Here $y = x + i\theta\sigma\bar{\theta}$. We see that W_α contains the gauge field strength $F_{\mu\nu}$. For this reason W_α is called the *field strength superfield* and plays a central role in construction supersymmetric gauge Lagrangians. As W_α involves the field strength, $W^2 \equiv W_\alpha W^\alpha$ will contain the kinetic terms. As W_α is a chiral superfield, it is clear we need an F-term Lagrangian. We therefore take

$$\begin{aligned} \frac{1}{2}\int d^2\theta W^\alpha W_\alpha = W^\alpha W_\alpha|_{\theta\theta} &= -i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \\ &\quad + \frac{D^2}{2} + \frac{i}{8}F^{\alpha\beta}F^{\gamma\delta}\epsilon_{\alpha\beta\gamma\delta}. \end{aligned} \tag{205}$$

To ensure the action is real we take

$$\int d^4x \, \frac{1}{4} \left(\int d^2\theta \, W^\alpha W_\alpha + \int d^2\bar{\theta} \, \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right) = \int d^4x \left(\frac{1}{2} D^2 - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - i\lambda\sigma^\mu \partial_\mu \bar{\lambda} \right). \tag{206}$$

These are the supersymmetric kinetic terms for an Abelian gauge field (and gaugino).

There is an easy (and important) generalisation of this. Suppose we have a chiral superfield

$$\Phi(y) = (f_R(y) + i f_I(y)) + \sqrt{2}\theta\psi + (\theta\theta)F. \tag{207}$$

Suppose also that $f_R(x) \neq 0$ and $f_I(x) \neq 0$, while $F = \psi = 0$. We call Φ the *gauge coupling superfield*, and it does not have to be dynamical.[3] We can now write down a Lagrangian

$$\begin{aligned} & \int d^4x \, \frac{1}{4} \left(\int d^2\theta \Phi W^\alpha W_\alpha + \int d^2\bar{\theta} \Phi^\dagger \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \right) \\ = & \int d^4x \left(\frac{1}{2} f_R D^2 - \frac{f_R}{4} F^{\mu\nu} F_{\mu\nu} - i f_R \lambda\sigma^\mu \partial_\mu \bar{\lambda} + \frac{f_I}{8} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} \right). \end{aligned} \tag{208}$$

Let us now write $f_R = \frac{1}{g^2}$ and $f_I = \theta_{YM}$, to then give

$$\int d^4x \left(\frac{1}{2g^2} D^2 - \frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \frac{i}{g^2} \lambda\sigma^\mu \partial_\mu \bar{\lambda} + \frac{\theta_{YM}}{8} \epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} \right). \tag{209}$$

This Lagrangian now has both a gauge coupling $\frac{1}{g^2}$ and also a topological θ_{YM} term $\epsilon_{\alpha\beta\gamma\delta} F^{\alpha\beta} F^{\gamma\delta}$. The values of both of these couplings are controlled by the vacuum expectatation value of the chiral multiplet Φ. This is an important statement and is the origin of several deep and powerful results. For an Abelian gauge theory, this however all seems slightly trivial - the topological term is always trivial, and without matter there is no intrinisic meaning to a $U(1)$ gauge coupling. The real power of this statement will come when we want to analyse the structure of non-Abelian supersymmetric gauge theories.

In general, if we have a term

$$\int d^2\theta \, f(\Phi) \left(W_\alpha W^\alpha + \text{c.c} \right), \tag{210}$$

then $f(\Phi)$ is called the *gauge kinetic function.* $\text{Re}(f)$ gives the physical gauge coupling and $\text{Im}(f)$ gives the theta angle.

Let us draw an important general conclusion from this: any theory which has *dynamical gauge couplings*, in which $f(\Phi)$ depends on a dynamical superfield Φ, also has a dynamical theta angle. What is the significance of a dynamica theta angle? The theta angle controls the phase of QCD instanton effects, and a dynamical theta angle is precisely equivalent to an axion.

[3] This means we do not have to include a kinetic term for it. If we feel squiffy about this, we can include a kinetic term for ϕ and also include an arbitrary large mass term in the superpotential $m\Phi\Phi$ so that the field is non-dynamical.

Put another way, any theory (for example string theory) which has both *supersymmetry* and *dynamical gauge couplings* automatically has axions in. This makes it clear that axions are an important generic expectation of the low energy physics arising from string theory.

Let us also note here another point that we will develop more fully later. Consider the action

$$\int d^2\theta \phi W_\alpha W^\alpha + \text{c.c} \tag{211}$$

Recall

$$\begin{aligned} W_\alpha &= -i\lambda_\alpha(y) + \ldots, \\ \Phi &= \phi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y). \end{aligned} \tag{212}$$

As a result, if $F_\Phi \neq 0$, then we have a term

$$\int d^4x \left(F_\Phi \lambda\lambda + \bar{F}_{\Phi^\dagger}\bar{\lambda}\bar{\lambda} \right), \tag{213}$$

which gives a Majorana mass for the gauginos. The significance of this, and what we learn from it, is that F-terms for the gauge kinetic function give gaugino masses. There is always a chiral gauge coupling superfield, and we can evaluate supersymmetry breaking gaugino masses by evaluating the F-term of this field.

We have so far discussed how to construct supersymmetric Lagrangians for Abelian gauge groups in supersymmetric theories. For Abelian theories there is another important term that can be added to the Lagrangian, the Fayet-Iliopoulos term,

$$\mathcal{L}_{FI} = \int d^2\theta d^2\bar{\theta} \xi V. \tag{214}$$

As $V = \ldots + \frac{1}{2}(\theta\theta)\left(\bar{\theta}\bar{\theta}\right)D(x)$, this gives a contribution to the Lagrangian of

$$L_{FI} = \int d^4x \, \frac{1}{2}\xi D(x). \tag{215}$$

This is gauge-invariant as under $V \to V + \Phi + \Phi^\dagger$, we have

$$\begin{aligned} \int d^4x \, \frac{1}{2}\xi D \quad &\to \quad \int d^4x \left(\frac{1}{2}\xi D + \frac{1}{4}\partial_\mu \partial^\mu \phi + \frac{1}{4}\partial_\mu \partial^\mu \phi^* \right) \\ &= \quad \int d^4x \frac{1}{2}\xi D(x), \end{aligned} \tag{216}$$

once we integrate by parts.

For Abelian theories, we have now enumerated the three basic kinds of terms involving vector fields:

1. Gauge kinetic terms
2. Gauge interactions of matter fields
3. Fayet-Iliopolous terms

We now combine these to see the full structure of the resulting Lagrangian and potential.

We first want to make a comment that we will return to later. As in regular field theory, we have a choice as to whether we put the gauge couplings in the kinetic terms or the interactions. This is the difference between

$$\frac{1}{g^2} F_{\mu\nu} F^{\mu\nu} + \phi^* A^\mu \partial_\mu \phi$$

and

$$F_{\mu\nu} F^{\mu\nu} + g \phi^* A^\mu \partial_\mu \phi.$$

Classically, these Lagrangians are related by the field redefinition $A_\mu \to g A_\mu$. Quantum mechanically, this field rescaling is anomalous and the two Lagrangians are different. We will discuss the quantum anomalies later.

We choose to put the gauge coupling in the kinetic term (this is more consistent with notions of holomorphy). So we have

$$\mathcal{L} = \frac{1}{4g^2} \left(\int d^2\theta W^\alpha W_\alpha + \text{c.c} \right) + \int d^4\theta \left(\xi V + \sum_i \Phi_i^\dagger e^{-qV} \Phi_i \right) \tag{217}$$

(Here q is the charge but contains no factors of the gauge coupling).

We can write this out in total, to get

$$\begin{aligned} \int d^4x \, \mathcal{L} \quad &= \quad \int d^4x \left(-\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu} - \frac{i}{g^2} \lambda \sigma^\mu \partial_\mu \bar{\lambda} + \frac{1}{2g^2} D^2 \right) \qquad (218) \\ &+ \left(F F^* - \partial_\mu \phi_i \partial^\mu \phi_i^* + i \partial_\mu \bar{\psi}_i \bar{\sigma}^\mu \psi_i \right) - A^\mu \left(\frac{1}{2} q_i \bar{\psi}_i \bar{\sigma}^\mu \psi_i + i q_i \phi_i^* \partial_\mu \phi_i \right) \\ &+ \frac{i}{\sqrt{2}} \left(q_i \phi_i \bar{\lambda} \bar{\psi}_i - q_i \phi_i^* \bar{\lambda} \psi_i \right) + \frac{1}{2} \left(q_i D - \frac{1}{2} q_i^2 v_\mu v^\mu \right) \phi_i^* \phi_i + \frac{1}{2} \xi D. \end{aligned}$$

Let's focus on the parts involving the non-dynamical D-term:

$$\int d^4x \frac{1}{2g^2} D^2 + \frac{D}{2} \left(q_i \phi_i^* \phi_i + \xi \right). \tag{219}$$

We solve to obtain $D = -\frac{g^2}{2} \left(\sum_i q_i \phi_i^* \phi_i + \xi \right)$, to give

$$V_D = \int d^4x \, \frac{g^2}{2} \left(\sum_i q_i |\phi_i|^2 + \xi \right)^2. \tag{220}$$

This is called the *D-term potential.* Together with the F-term potential, it is one of the two major sources of the potential in supersymmetric field theories. The D-term potential is

1. Associated to gauge fields, and proportional to the (gauge coupling)^2
2. Up to the FI terms, entirely determined by the gauge representations that fields are in.

3. Positive semi-definite: if $V_D > 0$ then supersymmetry is broken, as $\delta\lambda \sim \epsilon D$ for a vector superfield.

D-terms can also lead to supersymmetry breaking. Let us consider some simple models. The first case is the absolute simplest, corresponding to a U(1) gauge theory with no charged matter and a single FI term. In this case

$$\mathcal{L} = \int d^4x\, d^2\theta\, W_\alpha W^\alpha + \int d^4x\, d^2\theta d^2\bar{\theta}\xi V, \tag{221}$$

and so

$$V_D = \frac{g^2\xi^2}{2}.$$

This represents a constant flat potential with non-zero vacuum energy.

The second case is a slight generalisation to consider a U(1) theory, with an FI term and one chiral superfield of charge q). (However note that while consistent as a classical theory, quantum mechanically this theory is actually anomalous). The D-term potential is

$$V_D = \frac{g^2}{2}\left(q|\phi|^2 + \xi\right)^2. \tag{222}$$

There are then two cases. First, if q and ξ have opposite signs then a susy solution can be found and is located at $|\phi| = \sqrt{\frac{-\xi}{q}}$. In this case the gauge group is Higgsed due to the vev for ϕ. However, if q and ξ have identical signs then no susy solution exists. In this case the minimum has broken supersymmetry, with $\langle\phi\rangle = 0$ and a mass term for ϕ coming from its coupling to the FI term.

5.6 Non-Abelian Gauge Theories

Finally we wish to construct non-Abelian supersymmetric gauge theories. As before, the starting point is the vector multiplet V^a. The difference is that now this has to have components V^a for each generator of the gauge group. Chiral superfields now transform as

$$\begin{aligned} \Phi' &= e^{-i\Lambda}\Phi, \\ \Phi^{',\dagger} &= \Phi^\dagger e^{i\Lambda^\dagger}, \end{aligned} \tag{223}$$

with $\Lambda = T^a_{ij}\Lambda_a$ and T^a_{ij} the generator in the Φ representation. So if we naively generalise the Abelian coupling $\Phi^\dagger e^V \Phi$, then it transforms as

$$\Phi^\dagger e^V \Phi \to \Phi^\dagger e^{i\Lambda^\dagger} e^{V'} e^{-i\Lambda}\Phi. \tag{224}$$

We see that we can ensure a gauge invariant interaction by taking

$$e^{V'} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda}. \tag{225}$$

Here $V = V_a T^a_{ij}$ and so is matrix valued. An immediate worry here is that the definition (225) is in terms of a specific representation of the gauge group, and in this respect does not appear to be well defined. However recall that one case use the Hausdorff formula to simplify the products of exponentials. This involves multiple commutators of the exponents of the form $[A, B], [A, [A, B], \ldots$. These can all be simplified by use of the group commutation relations, $[T^a, T^b] = f^{abc}T^c$, until eventually we end up with a single generator and a relationship of the form

$$V^{'} = V_a^{'} T_{ij}^a, \tag{226}$$

where $V_a^{'}$ is now a direct function of the transformation parameters Λ and the original components V. The fact that these simplifications only require use of the group commutation relations means that this relationship is well defined and is independent of the representation originally used in the definition (225). As a result we can directly generalise the matter-gauge interaction to the case of non-Abelian theories, to give

$$\int d^4\theta \Phi^\dagger e^V \Phi, \tag{227}$$

with gauge transformations behaving as

$$\begin{aligned} \Phi &\rightarrow e^{i\Lambda}\Phi, \\ \Phi^\dagger &\rightarrow \Phi^\dagger e^{-i\Lambda^\dagger}, \\ e^{V^{'}} &\rightarrow e^{-i\Lambda^\dagger} e^V e^{i\Lambda}. \end{aligned} \tag{228}$$

In a similar vein we can also define the field strength superfield for non-Abelian theories,

$$W_\alpha = -\frac{1}{4}\bar{D}\bar{D}e^{-V} D_\alpha e^V. \tag{229}$$

The fact that $e^{-V}e^V = 1$ tells us that as

$$e^{V^{'}} = e^{-i\Lambda^\dagger} e^V e^{i\Lambda}, \tag{230}$$

then

$$e^{-V^{'}} = e^{-i\Lambda} e^{-V} e^{i\Lambda^\dagger}. \tag{231}$$

As a result W_α transforms as

$$\begin{aligned} W_\alpha &\rightarrow -\frac{1}{4}\bar{D}\bar{D}\left(e^{-i\Lambda}e^{-V}e^{i\Lambda^\dagger}\right) D_\alpha \left(e^{-i\Lambda^\dagger} e^V e^{i\Lambda}\right) \\ &= e^{-i\Lambda} W_\alpha e^{i\Lambda}, \end{aligned} \tag{232}$$

where we have used $D_\alpha \Lambda^\dagger = 0, \bar{D}\Lambda = 0$. It therefore follows that $\mathrm{Tr}(W_\alpha W^\alpha)$ is gauge-invariant, and so

$$\int d^2\theta \left(f(\Phi) W^\alpha W_\alpha + \mathrm{c.c}\right) \tag{233}$$

is a gauge invariant Lagrangian, which generalise the Abelian theory to a non-Abelian one.

The Lagrangian for a non-Abelian theory takes basically the same form as an Abelian one, with the obvious generalisations of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \rightarrow \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the presence of non-Abelian covariant derivatives.

One important area is the D-term. First, the coupling $\Phi^\dagger e^V \Phi$ now gives $D^{(a)}\Phi_i^\dagger T^a \Phi_i$ on taking the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component of V. As a result the D-term action looks like

$$\frac{(D^a)^2}{2g^2} + D^a \Phi_i^\dagger T^a \Phi_i, \tag{234}$$

and so

$$V_{D_a} = \frac{g^2}{2}\left(\sum_i \Phi_i^\dagger T_a \Phi_i\right)^2. \tag{235}$$

There is one contribution to the D-term potential for each generator of the gauge group, and so the total potential is given by

$$V_D = \frac{g^2}{2}\sum_a \left(\sum_i \phi_i^\dagger T_a \Phi_i\right)^2. \tag{236}$$

If there are multiple non-Abelian gauge groups, then we additionally need to sum over each group.

A further important feature of non-Abelian groups is that there are no Fayet-Iliopoulos terms for non-Abelian gauge groups, as the FI term

$$\int d^4x\, d^2\theta\, d^2\bar{\theta}\, \xi V \tag{237}$$

is not gauge invariant for non-Abelian theories.

It is a general result for non-Abelian theories that the F-terms give the structure of the vacuum. If we can solve the F-term equations, then it is always possible to solve the D-term equations.

To summarise then:

1. Chiral superfields give matter fields, and have Lagrangians described by a superpotential W and Kahler potential K, $\int d^2\theta W(\Phi)$ and $\int d^2\theta d^2\bar{\theta} K$. These generate an F-term potential $V_F = \sum_i \left(\frac{\partial W}{\partial \Phi}\right)^2$.
2. Vector superfields are used for gauge interactions, which come from an action

$$\int d^2\theta \frac{1}{g^2} W^\alpha W_\alpha + \int d^4\theta \Phi^\dagger e^V \Phi. \tag{238}$$

There is a D-term potential

$$V_D = \frac{g^2}{2}\sum_a \left(\sum_i \Phi_i^\dagger T_a \Phi_i + \xi\right)^2. \tag{239}$$

The FI term is only non-zero for Abelian gauge theories.

6 The MSSM

Supersymmetry may be a wonderful symmetry, but is it relevant to nature? This question has existed for a long time - the Supersymmetric Standard Model

Table 1: Field content of the SSM/MSSM

Field	$SU(3)$	$SU(2)_W$	$U(1)_Y$
$Q_{L,i=1,2,3}$	$\mathbf{3}$	$\mathbf{2}$	-1/6
$\bar{U}_{R,i=1,2,3}$	$\bar{\mathbf{3}}$	1	2/3
$\bar{D}_{R,i=1,2,3}$	$\bar{\mathbf{3}}$	1	-1/3
$L_{i=1,2,3}$	1	$\mathbf{2}$	1/2
$\bar{E}_{R,i=1,2,3}$	1	1	-1
$H_u = \begin{pmatrix} \hat{h}_u^+ \\ \hat{h}_u^0 \end{pmatrix}$	1	$\mathbf{2}$	-1/2
$H_d = \begin{pmatrix} \hat{h}_d^- \\ \hat{h}_d^0 \end{pmatrix}$	1	$\mathbf{2}$	1/2

was formulated by Fayet in 1977 and the first experimental bounds on supersymmetry appeared in 1978. If supersymmetry is present as a fundamental theory of nature, we have been waiting a long time for it.

Fortunately we have now developed enough formalism and technology to be able to construct the MSSM (Minimal Supersymmetric Standard Model). The MSSM is one of the principal search targets for the LHC experiments and represents the supersymmetrisation of the Standard Model with minimal extra ingredients.

The prime reason for the attractiveness of the MSSM is that it addresses the technical hierarchy problem of the Standard Model: what stabilises the Higgs mass against radiative corrections? [INSERT PICTURE] AS there is no mass term for fermions in the MSSM, there is also no mass term for bosons. Therefore at any scale where supersymmetry is a good symmetry, bosons must remain massless. The Higgs mass therefore cannot be parametrically larger than the supersymmnetry breaking scale. An alternative way of putting this is that mass terms are superpotential terms, and so in a supersymmetric theory are either present at tree level or not present at all. As a consequence, supersymmetric mass terms cannot be acquired at 1-loop and therefore masses in supersymmetry are radiatively stable.

What is the particle content of the MSSM? There are two basic types of multiplet present

1. Vector multiplets - there is one vector multiplet for each gauge generator of the $SU(3) \times SU(2) \times U(1)$ gauge group.
2. Chiral multiplets - there are chiral multiplets for each charged matter field present in the Standard Model.

Let us enumerate the chiral multiplets, together with their gauge charges. They are The immediate obvious point here is that the MSSM has *two* Higgs doublets in contrast to the Standard Model which only has one. Every other field is a simple susy-ification of the Standard Model matter content. There is an easy way to see why two Higgs doublets are necessary in supersymmetry. We recall that cancellation of gauge anomalies depends on the fermion sector of the theory.

In the Standard Model, the Higgs does not contribute to any anomalies because it is purely a scalar particle. However, once we promote the Higgs multiplet to a chiral multiplet, it contains a chiral fermion. This *does* contribute to gauge anomalies, and as for a theory with one Higgs doublet this is the only extra contribution to gauge anomalies (since the susy partners of the quarks and leptons are bosons), the resulting theory is anomalous and sick. However with two Higgs doublets the contibrution from each cancels, and the resulting theory is anomaly-free.

The MSSM is a renormalisable field theory. It has

- Constant (i.e. non field-dependent) gauge couplings, $f_a = \tau_a =$ constant.
- Canonical Kähler potential $K = \Phi^\dagger e^V \Phi$.
- Vanishing FI term, $\xi_{U(1)_Y} = 0$ (otherwise we would break electromagnetism *in vacuo* and give the photon a mass).

The gauge couplings of the MSSM have one important feature: extrapolated from the weak scale, they converge at $M_{GUT} \sim 2 \times 10^{16}$GeV. This famous result suggests that there is a higher unified structure present at the extremely small length scales that correspond to the GUT scale. However, this is not necessarily more than a suggestion - there is no direct evidence for GUTs and in particular proton decay, one of the characteristic signals of Grand Unified Theories, has not been observed.

Question: why is this plot meaningful? The significance of the plot is that three lines meet at a point, whereas two lines always meet at a point. However, there is no intrinsic notion of a gauge coupling for $U(1)$ gauge fields - the absence of a triple gauge boson vertex means that we can always rescale the gauge potential and thereby modify what we mean by the gauge coupling. So given that $U(1)_Y$ has no intrinsic coupling, why does the plot mean anything? (Somewhat elliptic hint: what is plotted is not $\frac{4\pi}{g_Y^2}$ but $\frac{5}{3} \times \frac{4\pi}{g_Y^2}$.)

Answer: the embedding of $U(1)_Y$ into $SU(5)$ provides a canonical normalisation of the $U(1)$.

What is the MSSM superpotential?

$$W = y_{u,ij} H_u Q_L^i \bar{U}_R^j + y_{d,ij} H_d Q_L^i \bar{D}_R^j + y_{L,ij} H_d L^j \bar{E}_R^j + \mu H_u H_d. \tag{240}$$

This has

- Yukawa couplings for up-type quarks (induced by H_u)
- Yukawa couplings for down-type quarks (induced by H_d)
- Yukawa couplings for leptons (induced by H_d)
- a supersymmetric Higgs mass term (the μ term)

The form of the superpotential is significantly restricted by holomorphy: in the Standard Model we could couple h or h^* to get Yukawa couplings, but this is forbidden by holomorphy (so no $H_d^* Q_L^i \bar{U}_R^j$ coupling for example). This is a second reason why there have to be two Higgs doublets in the supersymmetric Standard Model, as with only one Higgs doublet there would be no way of obtaining Yukawa couplings.

However, there are also extra terms which could in principle be added to the superpotential,

$$W_{\not{B}\not{L}} = \lambda_i \underbrace{LL\bar{E}_R}_{\Delta L=+1} + \lambda_2 \underbrace{LQ\bar{D}_R}_{\Delta L=+1} + \lambda_3 \underbrace{\bar{U}_R\bar{D}_R\bar{D}_R}_{\Delta B=-1} + \lambda_4 \underbrace{LH_2}_{\Delta L=+1} \qquad (241)$$

These terms have the crucial, and perhaps embarrassing feature, of violating either lepton or baryon number. This matters of course because both lepton and baryon number are good global symmetries of the Standard Model and are for all practical purposes exact. [4] These terms have to be highly restricted in order to avoid proton decay. If all four such terms $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ were non-zero then the proton can decay via the following process

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PROTON DECAY DIAGRAM

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The decay is mediated by a virtual squark $\tilde{d}$, and so the amplitude is suppressed by the squark propagator $\frac{1}{m_{\tilde{d}}^2}$ and the lifetime suppressed by $\frac{1}{m_{\tilde{d}}^4}$. One can do a full calculation, but it is simplest to observe that the only other dimensionful scale is the proton mass, and so the lifetime is

$$\tau_p \sim \left(\frac{m_{\tilde{d}}^4}{m_p^5}\right) \sim \left(\frac{1\text{TeV}}{1\text{GeV}}\right)^4 10^{-23s} \left(\frac{m_{\tilde{d}}}{1\text{TeV}}\right)^4 \sim \left(\frac{m_{\tilde{d}}}{1\text{TeV}}\right)^4 10^{-10} s. \qquad (242)$$

Such a lifetime is literally fatal, and we can see that this is entirely incompatible with the observed stability of the proton $\tau_p \gtrsim 10^{34}$years. To avoid this we need to forbid some or all of the operators in $W_{\not{B}\not{L}}$. The simplest option is to forbid all operators via R-parity. R-parity is defined by

$$R = (-1)^{3(B-L)+2S} = \begin{cases} +1 & \text{SM particles} \\ -1 & \text{SUSY particles} \end{cases} \qquad (243)$$

However this may be overkill (to forbid proton decay we only need to forbid one side of the diagram rather than both).

An implication of R parity is that the lightest supersymmetric particle (LSP) is stable, as its decays must involve another R-charged particle - which is not possible.

Let us also make a note on nomenclature: supersymmetric particles are named by adding 'o' (fermions) or 's' (scalars):

Quark	$\to$	Squark
Lepton	$\to$	Slepton
Gluino	$\to$	Gluino
W/Z	$\to$	Wino/Zino

[4] $B+L$ can be violated by electroweak sphaleron processes which are suppressed by $e^{-\frac{8\pi^2}{g_{YM}^2}}$. These may be important in the early universe when the temperature is comparable to the electroweak scale, but at the current time give a proton lifetime of $\tau \sim 10^{80}$years.

6.1 Yukawa Couplings

In the Standard Model we have the Yukawa couplings

$$y_t \phi \bar{q}_{L,t} t_R \quad \rightarrow \quad m_t = y_t < v >, \tag{244}$$

$$m_b = y_b < v > . \tag{245}$$

In the Standard Model there is then a single Higgs vev, and the different masses of the top and bottom quarks come from the different Yukawa couplings of the top and bottom quarks. The experimental fact that $m_t \sim 170\text{GeV}$ and $m_b \sim 4\text{GeV}$ then implies that $y_b/y_t \sim 1/40$, and so the bottom Yukawa coupling is intrinsically small.

What about the MSSM? In the MSSM there are two Higgs doublets, which can have separate vevs $\langle H_u \rangle$ and $\langle H_d \rangle$. The mass of the up-type quarks come from the vev of the up-type Higgs, and the mass of the down-type quarks come from the vev of the down-type Higgs. So

$$m_t = y_t \langle H_u \rangle, \tag{246}$$

$$m_b = y_b \langle H_d \rangle, \tag{247}$$

$$m_e = y_e \langle H_d \rangle. \tag{248}$$

As a result the up-type and down-type quarks obtain masses from *different* vevs. The importance of this is that the physical Higgs scalar - the one that gives mass to the W and Z bosons - is a combination of H_u and H_d. In the MSSM there are five Higgs scalars (8 from two complex doublets, of which 3 are eaten by the $SU(2)$ gauge bosons). The lightest CP even combination of these has very similar couplings to the Standard Model Higgs.

An important (and unknown) property of the MSSM is $\tan\beta$, which is the ratio of the up-type and down-type vevs,

$$\tan\beta = \frac{\langle H_u \rangle}{\langle H_d \rangle}. \tag{249}$$

The overall magnitude of the vev, $v = \sqrt{\langle H_u \rangle^2 + \langle H_d \rangle^2}$ is set by the mass of the W and Z bosons. However the extent to which this vev consists of $\langle H_u \rangle$ and $\langle H_d \rangle$ is undetermined, and this is what $\tan\beta$ specifies. The light Standard Model-like Higgs is then given by

$$h \sim \sin\beta H_u + \cos\beta H_d. \tag{250}$$

$\tan\beta$ is extremely important for the phenomenology of the MSSM, and gives one of the main differences between the couplings of a standard Model Higgs and an MSSM Higgs. The reason is that for large values of $\tan\beta$, $\langle H_d \rangle$ is actually rather small. In particular, for $\tan\beta \sim 40$, then $\langle H_u \rangle \sim 170\text{GeV}$ and $\langle H_d \rangle \sim 4\text{GeV}$. In this case the bottom Yukawa coupling y_b is close to unity, and as a result the bottom quark has an $\mathcal{O}(1))$ coupling to H_d. This is important in certain loop processes, as it means that compared to the Standard Model certain loop processes are $\tan\beta$ enhanced.

6.2 Supersymmetry Breaking

The 'supersymmetry' in MSSM is a misnomer: no superpartners have been observed and supersymmetry is necessarily broken. 'MSSM' refers to the supersymmetric Standard Model supplemented by *soft terms*. Soft terms are terms in the Lagrangian that explicitly break supersymmetry, are added to the Lagrangian of the supersymmetric Standard Model, but do *not* reintroduce quadratic divergences. In supergravity models, the soft terms arise from spontaneous breaking of supersymmetry and keeping the terms that survive in the $M_P \to \infty$ limit.

The possible form of soft terms was classified by Girardello and Grisaru in 1982. They are given by:

$$\text{Scalar mass terms} \quad m_Q^2 \tilde{Q}\tilde{Q}^* \tag{251}$$

$$\text{Gaugino Masses} \quad M_\lambda \lambda\lambda \tag{252}$$

$$\text{Trilinear scalar A-terms} \quad A\phi_i\phi_j\phi_k(\text{ where } \Phi_i\Phi_j\Phi_k \in W) \tag{253}$$

$$\text{Bilinear scalar B-terms} \quad B\phi_i\phi_j(\text{ where } \Phi_i\Phi_j \in W) \tag{254}$$

Note that all the soft terms are dimensionful - all dimension 4 (marginal) susy breaking operators are hard. This is as one would expect, as marginal operators are relevant on all distance scales, whereas relevant operators are important at low energies. Quadratic divergences are a feature of the high energy Lagrangian. Therefore marginal operators, which are important for high energy physics and modify the physics at all energy scales, are able to reintroduce quadratic divergences. In contrast, relevant operators - and all dimensionful operators are relevant - only affect long-distance physics and so cannot affect short-distance divergences.

The phenomenological theory called the MSSM is then defined by a Lagrangian

$$\mathcal{L} = \underbrace{\mathcal{L}_{SSM}}_{\text{Supersymmetric Standard Model}} + \underbrace{\mathcal{L}_{soft}}_{\text{soft breaking terms}} . \tag{255}$$

The soft breaking terms are expected to have masses around the weak scale if they are to be useful for addressing the hierarchy problem. The soft Lagrangian is an expression not of triumph but of failure: it is not a fundamental theory of supersymmetry breaking but instead a parametrisation of ignorance. Models of supersymmetry breaking aim at computing the parameters of the MSSM in terms of a deeper underlying theory.

Supersymmetric phenomenology is primarily about understanding how to generate the soft terms and the consequences of any given choice of soft terms. The rest of the Langrangian is entirely fixed by supersymmetry; the soft terms are what is left unknown. Supersymmetry is divine, whereas supersymmetry breaking infernal. All the many possible different models of supersymmetry are fundamentally different soft Lagrangians.

The big picture of supersymmetric model building is as follows:

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The starting point of a supersymmetric model is an explicitly supersymmetric theory. This theory may have one or many (meta)stable vacuum solutions, and one or many of these solutions may break supersymmetry. This means that the supercharges do not annihilate the vacuum. As the original theory was explicitly supersymmetric, the supersymmetry breaking is spontaneous. The full theory has two sectors:

- A *hidden sector*, in which the fields have non-zero F-terms. This is the sector of the theory in which supersymmetry is actually broken.
- The *visible sector*, which contains the MSSM fields. If the MSSM is extended to contain other fields with Standard Model charges and weak scale masses, these would normally also count as part of the visible sector. Supersymmetry breaking is communicated from the hidden sector to the visible sector by the mediator.

One natural question is: why not just have a single sector? Rather than have separate hidden and visible sectors, why not just break supersymmetry in the MSSM itself and be done with it? The problem with this is the supertrace formula of global supersymmetry,

$$\mathrm{STr}(M^2) = \sum_i m_{b,i}^2 - m_{f,i}^2 = 0. \tag{256}$$

Given the lightness of the MSSM fermions ($m_{u,d} \sim \mathcal{O}(5\mathrm{MeV})$), this implies that some of the squarks and sleptons must also be extremely light, and in fact sufficiently light to have already been observed.

Broadly speaking, there are two main ways of communicating supersymmetry breaking to the visible sector:

- *Gauge mediation*: supersymmetry breaking is communicated via renormalisable gauge interactions.
- *Gravity mediation*: This is a complete misnomer, and would better be called moduli mediation. Here supersymmetry breaking is communicated by intrinsically non-renormalisable operators that couple the visible sector and the hidden sector. Such operators are directly generated at the string scale and require an understanding of the string theory in order to study them properly.

It is useful to package both methods in a unified way based on how couplings depend on fields that break supersymmetry. The easiest case to consider is that of gaugino masses and A-terms. These turn out to be closely related to gauge and Yukawa couplings respectively. The appropriate terms in the superspace Lagrangians is

$$\int d^2\theta f_a(\Phi) W_\alpha W^\alpha + \int d^2\theta Y_{\alpha\beta\gamma}(\Phi) C^\alpha C^\beta C^\gamma \tag{257}$$

Now

$$\begin{aligned} W_\alpha &= \lambda_\alpha + \dots \quad (258) \\ C^\alpha &= \phi_i^\alpha + \theta \psi_i^\alpha + \dots \quad (259) \end{aligned}$$

If we expand the superfield $f_a(\Phi)$, we have

$$f_a(\Phi) = f_a(\phi) + \ldots + \theta^2 F_{f_a(\Phi)}.$$

Whichever superfield determines the gauge coupling, its F-term gives the gaugino mass. The same is true for the Yukawa couplings and A-terms: the lowest component of $Y_{\alpha\beta\gamma}(\Phi)$ gives the Yukawa couplings and the $(\theta\theta)$ component of $Y_{\alpha\beta\gamma}(\Phi)$ gives the A-term. What this tells us is that in a spontaneously broken supersymmetric theory, we can extract gaugino masses and A-terms by

1. Looking at how *gauge couplings* and *Yukawa couplings* depend on the fields that break supersymmetry.
2. Extracting the F-terms of these expressions.

To make this more explicit, let's consider gaugino masses. We have said these arise from the expression for gauge couplings. So what enters the value of gauge couplings? What determines that, say, $\alpha_{SU(3)} \sim 0.11$ when evaluated at M_Z?

We said earlier that gauge couplings appear to be unified at a high scale, and then run down from that high scale to lowe energy scales. What determines the couplings at this high scale? In string models, the gauge couplings at the string scale are typically set by a geometric feature of the compactification (for example, the size of a cycle or the value of the string coupling constant). In a low energy 4-dimensional theory, these parameters are themselves controlled by fields. For example, there is a dilaton field S such that $\mathrm{Re}(S) = \frac{1}{g_s}$ and a volume modulus T such that $\mathrm{Re}(T)$ is a measure of the volume of the compact space. These fields control the high-scale geometry and thereby also control the high-scale gauge couplings.

In this case the chiral superfields that control the size of these cycles enter the gauge couplings, for example as

$$f_a = \frac{T}{2\pi}. \quad (260)$$

(The linear dependence on T follows from the axionic nature of $\mathrm{Im}(T)$) Taking T to be a dimensionful field, we then must write

$$f_a = \frac{T}{2\pi M_X}, \quad (261)$$

where M_X is a dimensionful scale that could naturally be the Planck scale M_P or the string scale M_S. The classical gaugino Lagrangian then looks like

$$\frac{\mathrm{Re}(T)}{2\pi M_X} \bar{\lambda} \partial_\mu \lambda + \frac{F^T}{2\pi M_X} \lambda \lambda, \quad (262)$$

and so the physical gaugino mass is given by

$$M_{\lambda,physical} = \frac{F^T}{M_X \mathrm{Re}(f_a)}. \tag{263}$$

The suppression by the scale M_X appropriate to a non-renormalisable operator indicates that this is a gravity-mediated contribution. A general worry about non-renormalisable operators is that they cannot be computed at low energy - to compute them, you need to know the fundamental theory. This is true, and is an important motivation for understanding the fundamental theory better.

What else do gauge couplings depend on? We know that gauge couplings run at 1-loop, and that the beta function is determined by the number of light species.

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Using a Wilsonian approach,

$$\mathrm{Re}(f_a(\mu)) = \underbrace{\mathrm{Re}(f_a(\Lambda))}_{\text{string scale couplings}} + \underbrace{\beta \ln\left(\frac{\Lambda^2}{\mu^2}\right)}_{\text{running from scale } \Lambda \text{ to scale } \mu}. \tag{264}$$

As a superfield equation, the 1-loop gauge kinetic function is then

$$f_a = f_{a,0} + \beta_a \ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{265}$$

Now suppose there is a mass threshold present at a scale M. We can accommodate this threshold using a superpotential

$$W = MQ\bar{Q}, \tag{266}$$

which 'integrates out' a quark species at the scale M. In this case the gauge kinetic function would look like

$$f_a = f_{a,0} + \beta_1 \ln\left(\frac{\Lambda^2}{\mu^2}\right) + \beta_2 \ln\left(\frac{M^2}{\mu^2}\right). \tag{267}$$

We can rewrite this as

$$f_a(\mu) = f_{a,0} + (\beta_1 - \beta_2) \ln\left(\frac{\Lambda^2}{M^2}\right) + \beta_2 \ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{268}$$

When we introduced the mass threshold M, it was a fixed scale. Now let us instead regard it as the vev of a field X,

$$W = XQ\bar{Q}, \qquad \text{with } \langle X \rangle = M. \tag{269}$$

Allowing the vev of X to be arbitrary, we can extend (268) into a superspace equation,

$$f_a(\mu) = f_{a,0} + (\beta_1 - \beta_2) \ln\left(\frac{\Lambda^2}{M^2}\right) + \beta_2 \ln\left(\frac{\Lambda^2}{\mu^2}\right). \tag{270}$$

This encodes the dependence of the gauge couplings on X. However, as the gauge couplings depend on X, this also means that if $F^X \neq 0$ then gaugino masses are generated. Therefore: if we have

$$W = F_X X + XQ\bar{Q}, \qquad (\text{so } F_X = \frac{\partial W}{\partial X} \neq 0), \tag{271}$$

then

$$F^X \partial_X f_a = (\beta_1 - \beta_2) \frac{2F^X}{\langle X \rangle}, \tag{272}$$

giving the unnormalised gaugino masses. In this case, the gaugino mass was induced at 1-loop (as it comes from the difference in β-functions) and is parametrised by $\frac{F^X}{\langle X \rangle}$.

The scenario we have just described is that of *gauge mediation*: the generation of gaugino masses comes purely from renormalisable interactions and is a 1-loop effect. Let us think about the full expression for (Wilsonian) gauge couplings:

$$f_a(\mu) = \underbrace{f_{a,0}(\Phi)}_{\text{Tree level gauge couplings}} + \underbrace{(\Delta\beta)_i \ln\left(\frac{\Lambda^2}{X^2}\right)}_{\text{Mass threshols}} + \underbrace{\beta_{low} \ln\left(\frac{\Lambda_{UV}^2}{\mu^2}\right)}_{\text{Running gauge couplings}} \tag{273}$$

The entire expression should be understood as a superfield expression. We have labelled the different parts of this expression as to what conventional naes are applied to susy breaking originating in each of the different terms. This is really the absolute simplest way to understand the computation of gaugino masses,

1. What superfields do the gauge couplings depend on?
2. What are the F-terms of these fields?

There is a further subtlety to do with anomalies and the difference between Wilsonian and 1PI couplings, which we will return to later.

Everything we have just said for gauge couplings also applies to A-terms. We now replace the gauge couplings $f_a(\Phi) W_\alpha W^\alpha$ with the Yukawa couplings $Y_{ijk}(\Phi) C^i C^j C^k$. The F-term of $Y_{ijk}(\Phi)$ gives trilinear A-terms, and so there is a contibution to the A-terms given by

$$A_{ijk} \to F^I \partial_I Y_{ijk}(\Phi). \tag{274}$$

In general, there are additional terms in the expression for A-terms associated to the fact that the Yukawa couplings are not yet canonically normalised. The kinetic terms for the matter fields are given by $\int d^4\theta K_{ij}(\Phi, \bar{\Phi}) C^i C^{\bar{j}}$ and the *physical* Yukawa couplings depend on K_{ij}.

We can play a similar game for the matter fields. Consider a general Kähler potential

$$\int d^2\theta d^2\bar{\theta} Q^\dagger e^V Q K(\Phi, \bar{\Phi}). \tag{275}$$

Recall: the Kähler potential is not protected by holomorphy and is renormalised at all orders in perturbation theory.

If we view this as a superfield expansion, and consider the simplest possibility

$$\int d^2\theta d^2\bar{\theta} Q^\dagger e^V Q \frac{\Phi^\dagger \Phi}{\Lambda^2}, \tag{276}$$

then we see that for non-zero F^Φ that scalar masses arise from the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component of $\Phi^\dagger\Phi$:

$$\Phi^\dagger \Phi = \frac{(F^{\Phi,\dagger} F^\Phi)}{\Lambda^2} (\theta\theta)\left(\bar{\theta}\bar{\theta}\right) \implies m_Q^2 = \frac{|F|^2}{\Lambda^2}. \tag{277}$$

In fact, in general we see that the $(\theta\theta)(\bar{\theta}\bar{\theta})$ component of $K(\Phi, \bar{\Phi})$ will give rise to scalar masses.

We can view the kinetic terms in a similar way. The kinetic terms (effectively wavefunction normalisation) are modified at all orders in perturbation theory. We can write the Lagrangian at a given energy scale (i.e. in terms of the Wilsonian renormalisation group) as

$$\mathcal{L} = \int d^2\theta \left[f_a(\Phi)\right]_{1-loop} W_\alpha W^\alpha + \int d^2\theta d^2\bar{\theta} \left(Q^\dagger e^V Q\right) \left[K(\Phi, \Phi^\dagger)\right]_{\text{all loop}} + \int d^2\theta Y_{\alpha\beta\gamma}(\Phi) C^\alpha C^\beta C^\gamma. \tag{278}$$

Just as for the gauge couplings we can write the matter kinetic terms as

$$K(\Phi, \Phi^\dagger) = \underbrace{K_0(\Phi, \Phi^\dagger)}_{\text{tree level high scale}} + \underbrace{K_{1-loop}(\Phi, \Phi^\dagger)}_{\text{running due to thresholds}}. \tag{279}$$

Gravity mediation comes from the tree-level high scale kinetic terms (i.e. what is the normalisation at the string scale?). Gauge mediated soft terms come from running due to thresholds: squared masses arise at two loops.

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Let us make a couple of general comments about computing soft terms.

- Gravity mediated contributions need to be computed via the formalism of supergravity. There really is no option to this if you want to be doing a meaningful computation. For formulae, see Brignole, Ibanez and Munoz.
- Gauge mediated contributions are best computed *not* by working in a theory with broken supersymmetry and computing higher loop diagrams. Instead, it is better to work with supersymmetric theories and use techniques of analyticity in superspace.

Let us now discuss some general properties of gauge and gravity mediation. First of all we need the general expression

$$V = \sum_{i,j} K_{i\bar{j}} F^i F^{\bar{j}} - 3m_{3/2}^2 M_P^2 \tag{280}$$

$$= |F|^2 - 3m_{3/2}^2 M_P^2. \tag{281}$$

First, note that this implies that any consistent theory of supersymmetry breaking necessarily includes supergravity: we cannot just use global supersymmetry, even though it it useful for thinking about some of the issues.

For **gravity** mediation, we have

$$f \sim \frac{X}{M_P}, \qquad K \sim \frac{X^\dagger X}{M_P^2} Q^\dagger Q.$$

$$\implies M_\lambda \sim \frac{F^X}{M_P}, \qquad M_{Q^2} \sim \frac{|F^X|^2}{M_P^2}.$$

$$\implies F^X \sim (10^{11}\text{GeV})^2 \implies \frac{|F^X|^2}{M_P} \sim 1\text{TeV}, \text{ and } m_{3/2} \sim 1\text{TeV}.$$

For **gauge** mediation, we have

$$f \sim f_0 + \frac{\beta}{16\pi^2} \ln\left(\frac{M_P^2}{X^2}\right)$$

$$M_\lambda \sim \frac{\beta}{16\pi^2}\frac{F^X}{\langle X \rangle}, \qquad M_{Q^2} \sim \left(\frac{\beta}{16\pi^2}\right)^2 |\frac{F^X}{X}|^2.$$

We need $\frac{F^X}{\langle X \rangle} \sim 10^5\text{GeV}$. $\langle X \rangle$ is arbitrary. Limiting cases are

- $\langle X \rangle \sim 10^4\text{GeV} \to 10^9\text{GeV}^2 \sim (10^4\text{GeV})^2 \implies m_{3/2} \sim 0.1\text{eV}$.
- $\langle X \rangle \sim 10^{16}\text{GeV} \to F^X \sim 10^{21}\text{GeV}^2 \sim (10^{10}\text{GeV})^2 \implies m_{3/2} \sim (10\text{GeV})$.

An important consequence is that in gauge mediation the gravitino is the lightest supersymmetric particle.

Question: what is the strength of the coupling between the gravitino and matter? Why not just M_P?

Answer: Not so fast! Recall the super Higgs effect. The gravitino becomes massive (four degrees of freedom) by eating the goldstino. The longitudinal modes of the gravitino are those of the goldstino, and have couplings set by the goldstino.

So for gauge mediation, gravitino couplings are set by $\langle X \rangle$ rather than M_P: low-scale gauge mediation has LSP-gravitino couplings suppressed b $\frac{1}{(10\text{ TeV})}$ rather than $\frac{1}{M_P}$.

6.3 MSSM Phenomenology

MSSM phenomenology is determined by the pattern of soft masses at the weak scale. This is set by

1. Soft masses determined at the high (string) scale
2. RGE equations that evolve these terms down to the weak scale.

By now this evolution is highly automated and is carried out by various programs such as SoftSUSY. General features of the spectrum are

1. Coloured particles run up and tend to be heavy at low energies.
2. SU(2) charged particles tend to run down.
3. In the presence of large (as observed) top mass the H_u mass parameter runs negative and triggers radiative EWSB (Ibanez + Ross).

For practice gaining intuition as to the susy spectrum, go to kraml.web.cern.ch/kraml/comparison.

6.4 Flavour

One very important (possibly the most important) set of constraints on supersymmetry come from flavour physics. In fact these constraints are rather embarrassing, because flavour constraints show no evidence of any new physics at the TeV scale. For this to hold, the form of any new physics must take a highly constrained form. Often this form is governed by the assumption of *Minimal Flavour Violation*. The absence of any signs of new physics (or supersymmetry) through low energy flavour observables is known as the *susy flavour* and *susy CP* problems.

We will not give an exhaustive survey of all possible processes, but let us mention two. The first example is that of $K_0 - \bar{K}_0$ mixing, namely mixing between the neutral K_0 meson and its antipartner. $K_0 - \bar{K}_0$ mixing can occur in the Standard Model, but only as a loop level process that is also suppressed by CKM factors.

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It can also occur in the MSSM, where it is also a loop level process. In the case that flavour mixing is large, then the SUSY diagram can give a dominant contribution to $K\bar{K}$ mixing. However this is directly excluded because the experimental measurements of $K\bar{K}$ mixing are all consistent with no exotic contribution to the mixing.

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In the diagram, x refers to mass insertions that interconvert between $\tilde{s}$ and $\tilde{d}$ squarks. The dimensionless measure of the mixing is

$$\frac{m_{\tilde{s}}^2 - m_{\tilde{d}}^2}{m_{\tilde{s}} m_{\tilde{d}}}.$$

For large mixings - i.e. a susy mass spectrum that does not commute with the Yukawa flavour structure - this diagram gives a contribution to $K\bar{K}$ mixing that is far larger than the experimental limits. The absence of such contributions is known as the susy flavour problem. This problem is not restricted to the K meson system. For example, it also exists in systems of both D mesons (those involving charm quarks) and B mesons (involving b quarks), where oscillations of neutral mesons do not shown signs of large new contributions from Beyond-the-Standar-Model physics.

Another type of process leading to flavour constraints are those involving rare decays, for example the process $\mu \to e\gamma$. This is forbidden in the Standard Model as lepton flavour is a conserved quantity (we neglect the tiny effects of non-zero neutrino masses). However lepton flavour is only an accidental symmetry of the Standard Model and so can be badly violated in the MSSM with general flavour violating terms. An example of an MSSM diagram that leads to $\mu \to e\gamma$ is

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In the Standard Model the muon decays via weak processes in the decay mode $\mu \to e\nu_\mu\bar{\nu}_e$. As the weak interaction is highly suppressed at low energies, this gives a chance for susy contributions to have a measurable branching ratio. The measurement of the branching ratio $\mu \to e\gamma$ is one of the targets of the MEG experiment (for example), aiming to probe parameter space down to $BR(\mu to e\gamma) \lesssim 10^{-12}$.

The general danger in both of these diagrams, which can lead to large unobserved susy contributions, is that the flavour basis does not align with the mass basis. This imposes significant constraints on the soft terms of the MSSM. The basic solutions to these flavour problems are

1. Degeneracy : *universality* of soft scalar masses. If the soft masses are universal, then flavour rotations do not alter the structure of the soft mass matrix. In this case it is always possible to align the flavour and mass matrices, and so the dangerous flavour-violating processes are absent. This is the most commonly adopted solution of the supersymmetric flavour problem.

2. Alignment: in this case the structure of soft masses is not degenerate, but is aligned with respect to the Yukawas. Specifically, the soft scalar masses are diagonalised by the same rotations that diagonalise the Yukawas. In this case it is again possible to go to a basis where the flavour structure of the soft masses is identical to that of the Yukawas, and so there are no dangerous flavour violating processes.

3. Decoupling: the dangerous processes come from loops involving susy particles, and so involve factors of $\frac{1}{m_{\tilde{q}}^2}$. If susy particles have masses much larger than 1TeV, then such loops are small. For a flavour-anarchic spectrum, this requires that the masses are $m_{\tilde{Q}}^2 \sim (40\text{TeV})^2$. However, this is not a particularly good solution as this removes the motivation for supersymmetry in the first place.

An important general comment to make on the supersymmetric flavour constraints is that the constraints on the first two generations are much more severe than the constraints on the third generation. This should not be at all surprising - the first two generations are the lightest and involve cosmologically stable particles, and it is much easier to produce K mesons than it is to produce top quarks!

6.5 CP violation

A further source of constraints on the susy spectrum comes from the requirement that the susy spectrum does not introduce large new CP violating phases. CP violation is associated to complex phases in the Lagrangian and these can appear in several places, most notably in gaugino masses, A-terms and the μ-term.

The simplest solution to the CP problem is to require that all phases are universal: that is, there are no relative phases between the different gaugino masses and the A-terms are all proportional to the Yukawa couplings and with the same overall phases as the gaugino masses.

6.6 MSUGRA/CMSSM

One commonly used set of soft terms goes by the name of mSUGRA or CMSSM. Often these are taken to be interchangeable. However there are those who think that it is vital to distinguish correctly between the two. There are also those who think it is vital to distinguish between Marxist-Leninist-Trotskyist an Leninist-Marxist-Trotskyism. We shall neglect these important distinctions here. These soft terms are defined by universal high-scale boundary conditions at the GUT scale ($M_{GUT} \sim 2 \times 10^{16}$GeV). These involve the simplest choice of soft terms that satisfy the flavour and CP constraints.

1. *Universal* scalar masses m^2
2. *Universal* gaugino masses M with M real.
3. *Universal* A-terms $A_{\alpha\beta\gamma} = AY_{\alpha\beta\gamma}$ with A real.
4. Higgs sector terms μ and $B\mu$ are parametrised in a phenomenological fashion by $\tan\beta$ and sign(μ).

We shall discuss the last point in more detail later. Let us just say here that the soft parameters μ and $B\mu$, after minimising the Higgs potential, become equivalent to the Z mass m_Z, the ratio of the Higgs vevs $\langle v_u \rangle / \langle v_d \rangle$ and the sign of the μ parameter sign(μ). This spectrum is then evolved down to the weak scale to give the physical mass spectrum.

The universality of the scalar masses mean that the soft mass parameters for up-type squarks, down-type squarks, leptons and Higgs fields is taken to be the same at the GUT scale. The different low-scale value of the soft parameters comes purely from renormalisation group evolution of these parameters from short to long distances. The evolution starts at the GUT scale and is terminated at the weak scale. The reality of the gaugino masses and A terms is necessary to ensure the absence of large new sources of CP violation.

In total mSUGRA has 5 parameters: m^2, M_i, A, $\tan\beta$ and sign(μ). By now both the renormalisation group evolution of the soft terms (through programs such as softSUSY) and a scan over parameter space is highly automated.

The cosmological properties of supersymmetric spectra have also become highly automated with the development of programs such as microMEGAS. The R-parity conserving LSP is a WIMP dark matter candidate, and its thermal relic abundance can be computed automatically by a numerical solution of the Boltzmann equations. Such a particle can be searched for either via direct detection experiments:

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These search for wimp-nucleon scattering from WIMPs in the galactic dark matter halo. Another detection method is via indirect detection of annihilation signals from WIMPs that annihilate in the galactice centre. As the annihilation rate goes as the square of the density, the ideal region to search for such WIMPs are area with a large dark matter density and small baryonic densities. For these reason dwarf spheroidal galaxies are often used, as they have large mass-to-light ratios (which is a measure of the amount of dark matter present).

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Typical annihilation diagrams are shown above.

6.7 The Higgs Sector

The MSSM Higgs sector ($(1,2)_{-1/2} + (1,2)_{1/2}$) has a potential coming from

- F-terms $\mu H_u H_d$
- D-terms $g^2 \left(\sum_i \Phi_i^\dagger (T^a \Phi)_i\right)^2$
- soft terms $m_H^2 H_u^* H_u$ and $B\mu H_u H_d$

Putting these together, the tree level Higgs potential is

$$\begin{aligned} V_{Higgs} &= \left(m_{H_u}^2 + \mu^2\right)\left(|h_u^0|^2 + |h_u^+|^2\right) \\ &\quad + \left(m_{H_d}^2 + \mu^2\right)\left(|h_d^0|^2 + |h_d^-|^2\right) - B\mu\left(h_u^+ h_d^- + h_u^0 h_d^0 + \text{h.c}\right) \\ &\quad + \frac{g^2}{8}\left\{\left(|h_u^+|^2 - |h_u^0|^2 + |h_d^0|^2 - |h_d^-|^2\right)^2 + 4|h_u^+|^2|h_u^0|^2 + 4|h_d^0|^2|h_d^-|^2 \right. \\ &\quad \left. -4\left(h_u^{+,*} h_d^{-,*} h_u^0 h_d^0 + h_u^{0,*} h_d^{0,*} h_u^+ h_d^-\right)\right\} \\ &\quad + \frac{g'^2}{8}\left[|h_u^+|^2 + |h_u^0|^2 - |h_d^0|^2 - |h_d^-|^2\right]. \end{aligned} \tag{282}$$

The minimum of this potential has

$$\langle h_u^0 \rangle = \sqrt{2} v_u, \tag{283}$$
$$\langle h_d^0 \rangle = \sqrt{2} v_d, \tag{284}$$

where v_u and v_d are both real without loss of generality. As electric charge is preserved in the vacuum, there can be no mass mixing between charged and neutral Higgs fields. To obtain the charged Higgs mass matrix, we can therefore compute (the relevant fields are $h_u^+, h_u^{+,*}, h_d^-, h_d^{-,*}$)

$$M_{h^\pm}^2 = \frac{\partial^2 V}{\partial h^{u,\pm} \partial h^{d,\pm}}, \text{ with } h_u \to v_u, h_d \to v_d, \langle h_u^\pm \rangle = 0, \tag{285}$$

to obtain the charged Higgs mass matrix.

However we first study the neutral Higgs sector, putting $h^{u,\pm} = h^{d,\pm} = 0$. To do so we solve $\frac{\partial V}{\partial h_u^0} = \frac{\partial V}{\partial h_d^0} = 0$ (and conjugates). This gives the following equations:

$$\left(m_{H_u}^2 + \mu^2\right) h_u^0 - B\mu h_d^{0,*} + \frac{1}{4}\left(g^2 + g^{\prime,2}\right) h_u^0 \left(|h_u^0|^2 - |h_d^0|^2\right) = 0 \tag{286}$$
$$\left(m_{H_d}^2 + \mu^2\right) h_d^0 - B\mu h_u^{0,*} - \frac{1}{4}\left(g^2 + g^{\prime,2}\right) h_d^0 \left(|h_u^0|^2 - |h_d^0|^2\right) = 0. \tag{287}$$

These equations can be solved to give

$$B\mu = \frac{\left(m_{H_u}^2 + m_{H_d}^2 + 2\mu^2\right) \sin 2\beta}{2},$$
$$\mu^2 = \frac{m_{H_d}^2 - m_{H_u}^2 \tan^2\beta}{\left(\tan^2\beta - 1\right)} - \frac{M_Z^2}{2}. \tag{288}$$

Here M_Z is the Z mass, and we have used

$$M_Z^2 = \left(g^2 + g^{\prime,2}\right) \frac{v_u^2 + v_d^2}{2}. \tag{289}$$

This shows explicitly how $B\mu$ and μ can be related to M_Z and $\tan\beta$. Since M_Z is a known quantity, in practice we work backwards, and specify $\tan\beta$, using this to infer the value of μ and $B\mu$ at the high scale.

We can solve for the charged Higgs mass matrix to get

$$M_{h^\pm}^2 = \begin{pmatrix} B\mu \cot\beta + \frac{g^2}{2} v_d^2 & -B\mu - \frac{g^2}{2} v_u v_d \\ -B\mu - \frac{g^2}{2} v_u v_d & B\mu \tan\beta + \frac{g^2}{2} v_u^2 \end{pmatrix}. \tag{290}$$

The eigenvalues of this matrix are

$$m_{G^\pm}^2 = 0,$$
$$m_{H^\pm}^2 = B\mu\left(\cot\beta + \tan\beta\right) + M_W^2. \tag{291}$$

The zero eigenvalues correspond to the two Goldstone bosons of the broken SU(2) symmetry. These are eaten by the $SU(2)$ gauge bosons and become the longitudinal components of the $W^\pm$ gauge bosons.

Now consider the neutral sector. Again a decoupling occurs - as the Higgs sector is CP invariant, the real (CP even) and imaginary (CP odd) parts of the Higgs sector do not mix. For the imaginary parts we obtain

$$M^2_{h^I} = \frac{\partial^2 V}{\partial h^I_u \partial h^I_d} = \begin{pmatrix} B\mu \cot\beta & B\mu \\ B\mu & B\mu \tan\beta \end{pmatrix}, \tag{292}$$

with

$$\begin{aligned} m^2_{G_0} &= 0, \\ m^2_A &= B\mu\left(\cot\beta + \tan\beta\right). \end{aligned} \tag{293}$$

From this we see that (at tree level) $m^2_{H^\pm} = m^2_A + m^2_W$. The A particle is the *pseudoscalar* neutral Higgs boson, as it is odd under parity.

We can do the same trick for the neutral CP even Higgs scalar. In this case the mass matrix is

$$M^2_{h^0_R} = \begin{pmatrix} m^2_A \cos^2\beta + M^2_Z \sin^2\beta & -\left(m^2_A + M^2_Z\right)\sin\beta\cos\beta \\ -(m^2_A + m^2_Z)\sin\beta\cos\beta & m^2_A \sin^2\beta + M^2_Z \cos^2\beta \end{pmatrix}. \tag{294}$$

There are two neutral CP even Higgs scalars, denoted by h, H. From the above we find that

$$m^2_{h,H} = \frac{1}{2}\left[(m^2_A + m^2_Z) \pm \sqrt{(m^2_A + M^2_Z)^2 - 4m^2_A M^2_Z \cos^2 2\beta}\right]. \tag{295}$$

From this we obtain that

$$m_h \leq \quad m_A|\cos 2\beta| \quad \leq m_H, \tag{296}$$

$$m_h \leq \quad M_Z|\cos 2\beta| \quad \leq m_H. \tag{297}$$

The lightest MSSM Higgs suffers from the very important tree-level constraint

$$m_h < M_Z. \tag{298}$$

This constraint is important, as searches at LEP-II have ruled out the possibility of a Standard Model Higgs lighter than the Z mass!

Note that the physical propagating mass eigenstates

$$\underbrace{(G^\pm, G^0)}_{\text{Goldstones}}, \quad \underbrace{H^\pm}_{\text{charged Higgs}}, \quad \underbrace{A}_{\text{CP odd}}, \quad \underbrace{h, H}_{\text{CP even}}$$

are all related by mixing matrices to the original states. In fact,

$$\begin{pmatrix} G^+ \\ H^+ \end{pmatrix} = \begin{pmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} h^{-,*}_d \\ h^+_u \end{pmatrix}, \tag{299}$$

$$\begin{pmatrix} G^0 \\ A \end{pmatrix} = \begin{pmatrix} \sin\beta & -\cos\beta \\ \cos\beta & \sin\beta \end{pmatrix} \begin{pmatrix} h^0_{u,I} \\ h^0_{d,I} \end{pmatrix}, \tag{300}$$

$$\begin{pmatrix} h \\ H \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin \\ alpha & \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} h^0_{u,R} \\ h^0_{d,R} \end{pmatrix}. \tag{301}$$

Here α is a messy (although explicit) expression and we do not write it out in full.

General Comments about the Higgs Sector

- The Higgs sector preserves CP so long as both μ and $B\mu$ are real. This ensures that the minimum of the Higgs potential occurs for $\langle h_u \rangle$ and $\langle h_d \rangle$ both real and aligned, with no relative phase between them.

- Why is the tree-level Higgs mass sharply bounded by the Z mass? In the Standard Model, the Higgs mass can reach up to $\mathcal{O}(1000)$GeV. In the Standard Model, we had

$$V_H = -m^2|\phi|^2 + \lambda|\phi|^4, \tag{302}$$

  and m^2 and λ were both arbitrary. $\langle v \rangle = \sqrt{\frac{m^2}{2\lambda}}$ then determined the Z mass, and so there was one undetermined parameter left in the Higgs potential.

  In the MSSM, the mass squared terms are still arbitrary, as they come from the soft masses which are only determined by the high energy theory. However, the quartic term is no longer free: this is instead fixed by the D-term gauge interactions, which depend only on the (known) gauge couplings. So once we know the vev (i.e. M_Z) the mass of the Higgs is no longer arbitrary and is already determined. This is why there is a maximum mass of the MSSM Higgs: there is no leeway in the quartic term.

- Note however that if we add an additional gauge singlet scalar field to the theory, with a superpotential interaction

$$W = \lambda S H_u H_d, \tag{303}$$

  then this generates an F-term quartic coming from

$$V_F = |\frac{\partial W}{\partial S}|^2 = \lambda^2 |H_u|^2 |H_d|^2. \tag{304}$$

  This model is called the NMSSM and allows the Higgs mass constraints of the MSSM to be relaxed, due to the additional quartic contribution coming from F_S.

- Radiative corrections, however, matter. It is a very imporant feature of the MSSM that there are large radiative corrections to the tree-level Higgs potential. These originate from the large top mass and its $\mathcal{O}(1)$ Yukawa coupling. There are loops involving both tops and stops that renormalise both the quadratic and quartic Higgs couplings. Examples of these are shown in the diagrams below

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  FIGURE WITH FEYNMAN DIAGRAMS

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  In the decoupling limit ($m_H, m_A, m_{H^\pm} \gg m_h$), we have as a result

$$m^2_{h,1-loop} = m^2_{h,tree} + \frac{3g^2 m_t^4}{16\pi^2 M_W^2 \sin^2\beta} \ln\left[\left(1 + \frac{m^2_{\tilde{t}_L}}{m_t^2}\right)\left(1 + \frac{m^2_{\tilde{t}_R}}{m_t^2}\right)\right]. \tag{305}$$

Here g is the $SU(2)$ gauge coupling. Let us estimate this:

$$\frac{g^2}{4\pi}|_{SU(2)} \sim \frac{1}{30}, \qquad \frac{m_t^4}{M_W^2} \sim 16 M_Z^2, \qquad \ln\left[\frac{m_{\tilde{t}_L}^2 m_{\tilde{t}_R}^2}{m_t^4}\right] \sim 7 (m_{\tilde{t}} \sim 800\text{GeV}).$$

Putting this together gives (for these numbers) approximately

$$m_{h,1-loop}^2 = m_{h,tree}^2 + M_Z^2, \tag{306}$$

and so

$$m_{h,1-loop} \sim 130\text{GeV}.$$

While the detailed value of the 1-loop Higgs mass will depend on the value of the left and right-handed stops, as well as the measured value of the top mass, the clear lesson from this is that the 1-loop corrections are comparable to the tree-level terms and can raise the Higgs mass above the LEP exclusion limits - but not by much. One of the unambiguous predictions of the MSSM is a light Higgs with a mass not much larger than 130 GeV.

- In the large $\tan\beta$ regime, the Yukawa coupling of the bottom quark also becomes $\mathcal{O}(1)$, as does the tau Yukawa coupling. This is because down-type Yukawa couplings are given by $\frac{m_b}{m_t}\tan\beta$, and so approach unity in the large $\tan\beta$ regime. In this regime certain loop corrections can become large, as they rely on loops involving the bottom quark Yukawa coupling, and so are strongly enhanced in the large $\tan\beta$ limit.

  Certain processes that are $\tan\beta$ enhanced are $g_\mu - 2$, where $\Delta_{g-2,MSSM} \sim \tan\beta$ and the rare decay $B_s \to \mu^+\mu^-$, where $BR(B_s \to \mu^+\mu^-) \sim (\tan\beta)^6$.

7 Anomalies in Supersymmetric Theories

An anomaly is a classical symmetry of a theory that is not preserved quantum mechanically. The classic example of an anomaly is the axial anomaly of QCD, associated to independent rotations of the left and right-handed quarks.

$$Q_L \quad \to \quad e^{i\theta} Q_L, \tag{307}$$
$$Q_R \quad \to \quad e^{i\phi} Q_R. \tag{308}$$

The axial anomaly allows the decay $\pi_0 \to \gamma\gamma$ to proceed at a far more rapid rate than would otherwise be allowed (not suppressed by the scale of chiral symmetry breaking).

Key general features of anomalies are

- They depend on the fermionic sector of the theory and in particular on the number of massless fermions.

- They are 1-loop exact (this represents one of the few examples of a non-susy non-renormalisation theorem).

We are going to focus here specifically on *gauge coupling anomalies* in *supersymmetric gauge theories*.

Let us first of all state the problem, and then over the next few lectures go on to develop the solution. If we are interested in scattering amplitudes, it is natural to ask what the gauge coupling is at a scale μ. This can be probed by a scattering diagram such as

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The amplitude for this diagram should be given by

$$\mathcal{A} \sim \frac{g^2(t)}{t}. \tag{309}$$

The coupling depends on the scale of the process, and runs with energy scale. In particular, we are interested in the 1PI coupling, which represents the 'physical' gauge coupling (the coupling inferred from $2 \to 2$ scattering). This differs from the *holomorphic coupling* set by the gauge kinetic function, which is 1-loop exact, and is protected by the non-renormalisation theorems. Our aim here is to relate the two.

We want to relate the holomorphic action

$$\int d^2\theta \frac{1}{g_h^2} \left(W_\alpha(V_h) W^\alpha(V_h) + \text{h.c}\right) + \int d^2\theta d^2\bar{\theta} Z \Phi_h^\dagger e^V \Phi_h$$

to the physical action

$$\int d^4x d^2\theta \frac{1}{g_c^2} \left(W_\alpha \left(g_c V_c\right) W^\alpha(g_c V_c) + \text{h.c}\right) + \int d^4x d^2\theta d^2\bar{\theta} \left((\sqrt{Z}\Phi_h)^\dagger e^V (\sqrt{Z}\Phi_h) \right).$$

Classically, these are the same actions: we just define $g_c V_c = V_h$, $g_c = g_h$, $\Phi_c = \sqrt{Z}\Phi_h$ and then the two Lagrangians look the same. Classical physics is not affected by any field redefinitions: we can rescale fields and recover identical physics. This should ring alarm bells, as we know that many theories can have classical symmetries which do not hold quantum mechanically (for example QCD is classically scale-invariant but quantum-mechanically has running couplings). So as quantum physicists we should squint suspiciously at this field redefinition and suspect it will not hold quantum mechanically.

This is indeed correct. The technical issue is one that also occurs with the chiral anomaly. A quantum theory is defined not just by an action, but by a path integral:

$$\mathcal{A} = \int \mathcal{D}\Phi e^{-S(\Phi)}.$$

Even if $S(\Phi) = S(\Phi^{'})$, it does not follow that $D\Phi = D\Phi^{'}$. This invariance of the measure under transformations then leads to anomalous contributions to the gauge couplings. This is very similar to the Fujikawa derivation of the chiral anomaly as due to the (non)-invariance of the measure under chiral transformations.

$$\mathcal{D}(g_c V_c) \neq g_c \mathcal{D} V_c \tag{310}$$
$$\mathcal{D}\left(\sqrt{Z}\Phi\right) \neq \sqrt{Z}\mathcal{D}\Phi. \tag{311}$$

Before we deal with the more complicated case of supersymmetric gauge theories, it is helpful first to review the Fujikawa derivation of the chiral anomaly.

Consider the path integral for fermions in the presence of a gauge field,

$$Z = \int DA_\mu D\bar{\psi} D\psi \exp\left[-\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + i\int d^4x \bar{\psi}(i\not{D})\psi\right] \tag{312}$$

We use 4-dimensional Dirac spinors. The axial transformation is

$$\begin{aligned} \psi(x) &\rightarrow \&exp\left[i\alpha(x)\gamma^5\right]\psi(x), \\ \bar{\psi}(x) &\rightarrow \bar{\psi}(x)\exp\left[-i\alpha(x)\gamma^5\right]. \end{aligned} \tag{313}$$

For an infinitesimal transformation, the Lagrangian integrand behaves as

$$\mathcal{L} \rightarrow \mathcal{L} - \partial_\mu \alpha(x)\bar{\psi}\gamma^\mu\gamma_5\psi, \tag{314}$$

and so is invariant for constant α. It also naively appears that

$$D\bar{\psi} D\psi \rightarrow D\bar{\psi} D\psi, \tag{315}$$

under axial rotations, and so naively it appears that the entire functional integral is invariant. However this last relation is only a naive one and in fact is seen to be false under a careful analysis.

What's the problem? First, we should be careful about what we mean by $D\psi$. $\psi(x)$ is a field, and the actual modes that we can quantise are the eigenmodes o ψ. So we should first of all expand $\psi(x)$ and $\bar{\psi}(x)$ in terms of eigenmodes of the Dirac equation,

$$\begin{aligned} \psi(x) &= \sum_n a_n \phi_n(x), \\ \bar{\psi}(x) &= \sum_n \phi_n^\dagger(x)\bar{b}_n. \end{aligned} \tag{316}$$

Orthogonality here means that $\int \phi_n^\dagger(x)\phi_m(x)d^4x = \delta_{n,m}$. The eigenmodes $\phi_n(x)$ are defined by

$$\begin{aligned} (i\not{D})\phi_n(x) &= \lambda_n\phi_n(x). \\ \phi_n^\dagger(x)(i\not{D}) &= \lambda_n\phi_n^\dagger(x). \end{aligned} \tag{317}$$

Here D is the covariant derivative, $D_\mu = \partial_\mu + ieA_\mu$ (for an Abelian theory). In general we do not need to know the explicit form of the eignefunctions, although for zero gauge field ($A_\mu = 0$) these are just the standard plane wave eigenfunctions. The degrees of freedom are the coefficients of the eigenmodes, and so having carried out the decomposition we can write the functional integral as

$$\mathcal{D}\psi\mathcal{D}\bar{\psi} = \prod_m d\bar{b}_m da_m \tag{318}$$

This is then a sum over all possible coefficients of the eigenfuntions, and makes the general form of the functional integral $D\psi$ explicit.

An infinitesimal chiral transformation takes $\psi(x) \to \psi^{'}(x) = (1+i\alpha(x)\gamma^5)\psi(x)$. Obviously this also gives an infinitesimal transformation on the expansion coefficients $a_n, \bar{b}_m$.

$$\begin{aligned} a^{'}_m &= \sum_n \int d^4x \phi^{\dagger}_m(x) \left[(1 + i\alpha(x)\gamma^5) a_n \phi_n(x)\right] \\ &= \sum_n \left(\delta_{mn} + C_{mn}\right) a_n. \end{aligned} \tag{319}$$

where

$$C_{mn} = \int d^4x \phi^{\dagger}_m i\alpha(x)\gamma^5 \phi_n(x).$$

This can be viewed as a coordinate change on the 'coordinates' $a_m, \bar{b}_m$. As these are Grassmann coordinates, the measure therefore transforms as

$$D\bar{\psi}D\psi \to D\bar{\psi}D\psi \left[\det(1 + C)\right]^{-2}. \tag{320}$$

(as there is one transformation for $D\psi$ and one for $D\bar{\psi}$). So we need to evaluate

$$\begin{aligned} J = \det(1 + C) &= \exp\left[\mathrm{tr}\log(1 + C)\right] \\ &= \exp\left[\sum_n C_{nn} + \ldots + \mathcal{O}(C^2)\right]. \end{aligned} \tag{321}$$

For infinitesimal α we only need the first term in this expression, and so have

$$\ln J = \mathrm{tr}(C) = i \int d^4x \alpha(x) \sum_n \phi^{\dagger}_n(x)\gamma^5\phi_n(x). \tag{322}$$

Note that

- This is in some sense undefined, as it sums over an infinite tower of modes and therefore needs to be regulated.
- The sum basically counts 'number of left-moving solutions of the Dirac equation' minus 'number of right-moving solutions of the Dirac equation'. We can see this by the presence of γ_5, which separates out left- and right-handed solutions with a relative minus sign.
- If we are smart, we now understand the answer. Massive modes always pair up into an equal number of chiral and antichiral states, and so cannot contribute. Massless modes can have a net chirality which is given by an index theorem and is determined by the Pontryagin number of the gauge background.

Let's see how this arises explicitly. We insert a UV regulator that cuts off the massive modes. The natural choice for a UV regulator is to de-weight high energy modes exponentaially with their mass. We can write

$$\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x) = \lim_{M\to\infty}\sum_n \phi_n^\dagger(x)\gamma^5\phi_n(x)e^{-\lambda_n^2/M^2}$$
$$= \lim_{M\to\infty} \mathrm{Tr}\langle\phi_n|\gamma^5 e^{\left(i\not{D}\right)^2/M^2}|\phi_n\rangle$$
$$= \lim_{M\to\infty} \mathrm{Tr}\langle x|\gamma^5 e^{\left(i\not{D}\right)^2/M^2}|x\rangle. \quad (323)$$

Now $(i\not{D})^2 = -D^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$ (for a background field). As we have a (regulated) trace over all possible states, we can evaluate this in any basis we choose. Let's choose the simplest possible basis of states, that of plane-waves.

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{ik\cdot x}.$$

So we have

$$\lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4} e^{-ik\cdot x}\mathrm{Tr}\left(\gamma^5 e^{(-D_\mu D^\mu + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu})^2/M^2}\right) e^{ik\cdot x}$$

We can move $e^{ik\cdot x}$ through and replace D_μ by k_μ, to get

$$\lim_{M\to\infty}\int \frac{d^4k}{(2\pi)^4}\mathrm{Tr}\left(\gamma^5 \exp\left[\frac{(-k_\mu k^\mu + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu})}{M^2}\right]\right).$$

To get a non-zero trace over all four polarisation vectors, we need to bring down four gamma matrices, and so we have

$$\lim_{M\to\infty} \mathrm{Tr}\left(\gamma_5[\gamma^\mu,\gamma^\nu]F_{\mu\nu}\right)^2 \frac{1}{(2M^2)^2}\frac{1}{2!}\int \frac{d^4k}{(2\pi)^4}e^{-k^2/M^2}.$$

The $\int d^4k$ integral now cancels the residual factor of M^{-4}, leaving an overall finite answer of

$$\int d^4x \frac{-1}{8\pi^2}F_{\mu\nu}(*F)^{\mu\nu} \quad (324)$$

This then implies that the integrand measure transforms under chiral transformations as

$$\int DA_\mu D\bar{\psi}D\psi \to \int DA_\mu D\left(\bar{\psi}e^{-i\alpha\gamma_5}\right)D(e^{i\alpha\gamma_5}\psi) = DA_\mu D\bar{\psi}D\psi\left(1 + \int d^4x\frac{-i\alpha}{8\pi^2}F_{\mu\nu}(*F)^{\mu\nu}\right).$$

The integrand measure is therefore *not invariant* under chiral transformations. This actually makes perfect sense: the path integral counts (number of left movers) minus (number of right movers). This difference is precisely accounted for by the instanton number. In non-zero instanton backgrounds, there really are more left-movers than right-movers and axial symmetry is not a symmetry. This accounts for the topolgical form of the chiral anomaly. Instanton backgrounds do have different numbers of left and right-movers, and so have an intrinsic notion of chirality - it is therefore natural that the measure should fail to be invariant in such backgrounds.

7.1 The Konishi Anomaly

Let us now generalise this to the case of supersymmetric field theories. Our first generalisation is to the Konishi anomaly, which is the supersymmetrisation of the chiral anomaly.

The path integral of a chiral multiplet in a gauge field background is

$$\int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}F\mathcal{D}\bar{\phi}\mathcal{D}\bar{\psi}\mathcal{D}\bar{F}\exp\left[-\int d^4x|D\phi|^2 + \bar{\psi}\not{D}\psi - \bar{F}F\right].$$

We want to perform a general rescaling $\Phi = e^{\alpha}\Phi^{'}$. Naively,

$$\mathcal{D}\Phi = \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}F = \mathcal{D}\phi^{'}\left(\det e^{\alpha}\right)\mathcal{D}\psi^{'}\left(\det e^{\alpha}\right)^{-2}\mathcal{D}F\left(\det e^{\alpha}\right) \equiv \mathcal{D}\Phi^{'}J.$$

Here the negative powers of the determinant are due to the Grassmanian nature of the fermions. We now want to define and regularise the Jacobian. We proceed in the same way as before.

First, we expand the scalar field ϕ in a basis of eigenstates.

$$\phi(x) = \sum_n \lambda_n\phi_n(x), \text{ so that}\mathcal{D}\phi = \prod_n d\lambda_n.$$

The redefined field is then expanded in a modified basis of eigenstates,

$$e^{\alpha}\phi(x) = \sum\lambda_n^{'}\phi_n(x), \qquad \text{with } \lambda_n^{'} = \int\phi_n^{\dagger}(x)e^{\alpha}\phi(x) \equiv C_{nm}\lambda_m.$$

Here

$$C_{nm} = \int \phi_n^{\dagger}(x)e^{\alpha(x)}\phi_m(x)\mathrm{dx}.$$

As a result,

$$\prod d\lambda^{'} = (\det C)\prod d\lambda$$

and

$$\begin{aligned}(\det c) &= \exp\left[\mathrm{tr}\log\int\phi_n^{\dagger}(x)e^{\alpha(x)}\phi_m(x)\right] &(325)\\ &= \exp\left[\mathrm{tr}\int\phi_n^{\dagger}(x)\alpha(x)\phi_m(x)\right] &(326)\\ &= \exp\left[\sum_n\int\alpha(x)\phi_n^{\dagger}(x)\phi_n(x)\right]. &(327)\end{aligned}$$

As a consequence we have

$$\ln J = \int d^4x\alpha(x)\sum_n\phi_n^{\dagger}(x)\phi_n(x). \qquad (328)$$

We regulate this as before by cutting off the high momentum modes, which is most easily carried out by using an exponential cutoff.

$$\begin{aligned}\sum_n\phi_n^{\dagger}(x)\phi_n(x) &\rightarrow \sum_n e^{-\lambda_n^2 t}\phi_n^{\dagger}(x)\phi_n(x) &(329)\\ &= \mathrm{Tr}_{\phi}e^{t(D_{\mu})^2}. &(330)\end{aligned}$$

We have a similar effect for both the fermionic and F terms, so we have

$$\ln J = \alpha \left(\mathrm{Tr}_\phi e^{t(D_\mu)^2} - \mathrm{Tr}_\psi e^{t(\not{D})^2} + \mathrm{Tr}_F e^{t(D_\mu)^2} \right). \tag{331}$$

We are now going to evaluate this explicitly. We work in Abelian gauge theory and impose a constant electromagnetic field background (this does not significantly affect anything).

$$F_{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}. \tag{332}$$

Then $D_\mu = \partial_\mu + iA_\mu = i(-i\partial_\mu + A_\mu)$. We now make a gauge choice

$$A_1 = -\frac{Ex_2}{2}, A_2 = \frac{Ex_1}{2}, A_3 = -\frac{Bx_4}{2}, A_4 = \frac{Bx_3}{2}.$$

We also write $p_\mu = -i\partial_\mu$. Then

$$\mathrm{Tr} e^{t(D_\mu)^2} = \mathrm{Tr}\exp\left[-t\left(\left(p_1 - \frac{Ex_2}{2}\right)^2 + \left(p_2 + \frac{Ex_1}{2}\right)^2 + \left(p_3 - \frac{Bx_4}{2}\right)^2 + \left(p_4 + \frac{Bx_3}{2}\right)^2 \right)\right]. \tag{333}$$

(Note that p_μ here is an operator). We now define

$$\begin{aligned} H(E) &= \left(p_1 - \frac{Ex_2}{2}\right)^2 + \left(p_2 + \frac{Ex_1}{2}\right)^2 \\ &= p_1^2 + p_2^2 + \left(\frac{E}{2}\right)^2 (x_1^2 + x_2^2) - E(p_1 x_2 - p_2 x_1), \end{aligned} \tag{334}$$

and likewise

$$H(B) = \left(p_3 - \frac{Bx_4}{2}\right)^2 + \left(p_4 + \frac{Bx_3}{2}\right)^2. \tag{335}$$

Now, as (x_1, x_2, p_1, p_2) are simulataneously diagonalisable, we can therefore split

$$\begin{aligned} \mathrm{Tr} e^{t(D_\mu)^2} &= \mathrm{Tr} e^{-tH(E)H(B)} \\ &= \mathrm{Tr}_{(x_1,x_2)} e^{-tH(E)} \mathrm{Tr}_{(x_3,x_4)} e^{-tH(B)}. \end{aligned} \tag{336}$$

To evaluate the traces we take

$$p_\mu = \sqrt{\frac{E}{2}} \left(\frac{a_\mu - a_\mu^\dagger}{i\sqrt{2}} \right), \qquad x_\mu = \sqrt{\frac{2}{E}} \left(\frac{a_\mu + a_\mu^\dagger}{\sqrt{2}} \right).$$

We can then write

$$\begin{aligned} H(E) &= \frac{E}{2} \times -\frac{1}{2} \left[(a_1 - a_1^\dagger)^2 + (a_2 - a_2^\dagger)^2 \right] + \frac{E}{2} \times \frac{1}{2} \left[(a_1 + a_1^\dagger)^2 + (a_2 + a_2^\dagger)^2 \right] - \frac{E}{2i} \Big((a_1 - a_1^\dagger)(a_2 + a_2^\dagger) - \\ &= \frac{E}{4} \left[2(a_1 a_1^\dagger + a_1^\dagger a_1) + 2(a_2^\dagger a_2 + a_2^\dagger a_2) + 2i(2a_1 a_2^\dagger - 2a_1^\dagger a_2) \right] \\ &= \frac{E}{2} \left[(a_1^\dagger a_1 + a_1 a_1^\dagger) + (a_2^\dagger a_2 + a_2^\dagger a_2) + 2i(a_1 a_2^\dagger - a_1^\dagger a_2) \right] \\ &= \frac{E}{2} \left[2a_1^\dagger a_1 + 2a_2^\dagger a_2 + [a_1, a_1^\dagger] + [a_2, a_2^\dagger] + 2i(a_1 a_2^\dagger - a_1^\dagger a_2) \right]. \end{aligned}$$

Now define $a_L = \frac{a_1 - i a_2}{\sqrt{2}}$, $a_R = \frac{a_1 + i a_2}{\sqrt{2}}$. This then gives

$$a_L^\dagger a_L = (a_1^\dagger + i a_2^\dagger)(a_1 - i a_2)$$
$$= \frac{1}{2}\left(a_1^\dagger a_1 + a_2^\dagger a_2 - i(a_1^\dagger a_2 - a_2^\dagger a_1)\right). \tag{338}$$

So then

$$H(E) = E\left(2a_L^\dagger a_L + \frac{[a_1, a_1^\dagger] + [a_2, a_2^\dagger]}{2}\right),$$

and so

$$H(E) = E(2a_L^\dagger a_L + 1).$$

a_L still has the canonical harmonic oscillator commutation relations, and so

$$\mathrm{Tr} e^{-tH(E)} = \sum_{n_L, n_R} e^{-tE(2n_L+1)}.$$

The sum over both n_L and n_R reflects the fact that we need to sum over both a_L and a_R excitations. The fact that this sum diverges reflects a divergence due to infinite space-time volume. Let's make this more precise. We go to finite volume (radius L) and only want to sum over modes with

$$\langle n_L, n_R | x^2 + y^2 | n_L, n_R \rangle < L^2$$

Now

$$\begin{aligned} x^2 + y^2 &= \frac{2}{2E}\left(\left(a_1 + a_1^\dagger\right)^2 + \left(a_2 + a_2^\dagger\right)^2\right) \\ &= \frac{1}{2E}\left(2\left(a_L a_L^\dagger + a_L^\dagger a_L\right) + 2\left(a_R a_R^\dagger + a_R^\dagger a_R\right)\right). \\ &= \frac{1}{E}\left(2n_L + 2n_R + 2\right). \end{aligned} \tag{339}$$

As a result $x^2 + y^2 = \frac{2}{E}(n_L + n_R + 1)$, and so the condition that $x^2 + y^2 < L^2$ translates to the condition that $\frac{2}{E}(n_L + n_R + 1) < L^2$. As a a result we obtain the bound

$$n_R < \frac{EL^2}{2} - n_L - 1. \tag{340}$$

Our regularised sum is now

$$\begin{aligned} \mathrm{Tr} e^{-tH(E)} &= \sum_{n_L, n_R} e^{-tE(2n_L+1)} \\ &= \sum_{n_L}\left(\frac{EL^2}{2} - n_L - 1\right) e^{-tE(2n_L+1)}. \end{aligned} \tag{341}$$

To leading order in L^2 (i.e. at large volume), we have

$$\begin{aligned} \frac{EL^2}{2} e^{-tE}\left(1 + e^{-2tE} + (e^{-2tE})^2 + \ldots\right) &= \frac{EL^2}{2}\frac{e^{-tE}}{1 - e^{-2tE}} + \ldots \\ &= \frac{(\pi L^2)E}{4\pi}\frac{1}{\sinh(tE)}. \end{aligned} \tag{342}$$

Going to infinite volume, we therefore have

$$\mathrm{Tr}e^{-tH(E)} = \frac{1}{4\pi}\int dx_0 dx_1 \frac{E}{\sinh tE}. \tag{343}$$

Performing the same calculation for $H(B)$, we then obtain

$$\mathrm{Tr}e^{t(D_\mu)^2} = \frac{1}{16\pi^2}\int d^4x \frac{EB}{\sinh tE \sinh tB} \tag{344}$$

This diverges as we take the regulator $t \to 0$ (as we would expect). As this is just a scalar field, we also get the same result for the F-term. So at this point we now have two components of the anomalous Jacobian,

$$\ln J = \alpha \left(\underbrace{\mathrm{Tr}_\phi e^{t(D_\mu)^2}}_{\text{yes}} - \mathrm{Tr}_\psi e^{t\not{D}^2} + \underbrace{\mathrm{Tr}_F e^{t(D_\mu)^2}}_{\text{yes}} \right). \tag{345}$$

We next need the regulated Dirac determinant. The Dirac operator is

$$\begin{aligned} (\not{D})^2 &= D_\mu^2 - \frac{i}{4}[\gamma^\mu, \gamma^\nu]F_{\mu\nu} \\ &= (D_\mu)^2 + \begin{pmatrix} (E+B)\sigma^3 & 0 \\ 0 & (-E+B)\sigma^3 \end{pmatrix}. \end{aligned} \tag{346}$$

Any fermionic state can be written as a linear combination of the basic wave-functions

$$\psi(x) \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad \psi(x) \otimes \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We can therefore also factorise

$$\begin{aligned} \mathrm{Tr}_{L,R} e^{t(\not{D})^2} &= \mathrm{Tr}_{L,R} e^{t(D_\mu)^2} e^{\begin{pmatrix} (E+B)\sigma^3 & 0 \\ 0 & (-E+B)\sigma^3 \end{pmatrix}} \\ &= \mathrm{Tr}e^{t(D_\mu)^2)}\mathrm{Tr}e^{t(\pm E+B)}, \end{aligned} \tag{347}$$

where $+$ applies for left-handed modes and $-$ applies for right-handed modes. Now,

$$\begin{aligned} \mathrm{Tr}e^{t(\pm E+B)\sigma^3} &= e^{t(\pm E+B)} + e^{-t(\pm E+B)} \\ &= 2\cosh t(E \pm B). \end{aligned} \tag{348}$$

So the fermionic heat kernel is then

$$\begin{aligned} \mathrm{Tr}_{L,R} e^{t\not{D}^2} &= \mathrm{Tr}e^{t(D_\mu)^2}\mathrm{Tr}e^{t(\pm E+B)\sigma^3} \\ &= \frac{1}{16\pi^2}\int d^4x \frac{EB \times 2\cosh t(E \pm B)}{\sinh Et \sinh Bt}. \end{aligned} \tag{349}$$

We can now sum (left-handed fermion)

$$\ln J = \alpha \left(\mathrm{Tr}_\phi e^{t(D_\mu)^2} - \mathrm{Tr}_{\psi_L} e^{t\not{D}^2} + \mathrm{Tr}_F e^{t(D_\mu)^2} \right) \tag{350}$$

$$= \frac{\alpha}{16\pi^2}\int d^4x EB \frac{2 - 2\cosh t(E+B)}{\sinh Et \sinh Bt}. \tag{351}$$

and likewise (right-handed fermion)

$$\ln \bar{J} = \alpha \left(\mathrm{Tr}_\phi e^{t(D_\mu)^2} - \mathrm{Tr}_{\psi_R} e^{t\not{D}^2} + \mathrm{Tr}_F e^{t(D_\mu)^2} \right) \tag{352}$$

$$= \frac{\alpha^*}{16\pi^2} \int d^4x EB \frac{2 - 2\cosh t(E-B)}{\sinh Et \sinh Bt}. \tag{353}$$

Let's look at evaluating this for small t.

$$\cosh \epsilon = \frac{e^\epsilon + e^{-\epsilon}}{2} = 1 + \frac{1}{2}\left(\frac{\epsilon^2}{2} + \frac{\epsilon^2}{2}\right) = 1 + \frac{\epsilon^2}{2}.$$

and so

$$\begin{aligned} 2 - 2\cosh t(E-B) &= 2 - 2\left(1 + \frac{t^2(E-B)^2}{2}\right) \\ &= -t^2(E-B)^2. \end{aligned} \tag{354}$$

Therefore

$$\begin{aligned} \frac{(2 - 2\cosh[t(E \pm B)])EB}{\sinh Et \sinh Bt} &\sim -\frac{t^2(E-B)^2 EB}{EBt^2} \\ &\sim -(E \pm B)^2. \end{aligned} \tag{355}$$

So, including the cutoff and regulating we emerge with

$$\ln J = \frac{\alpha}{16\pi^2} \int d^4x - (E+B)^2.$$

This is now a finite and well-defined answer. Now,

$$F_{\mu\nu}F^{\mu\nu} = -2(E^2 + B^2),$$

$$F_{\mu\nu}(*F)^{\mu\nu} = -4EB.$$

So

$$(E+B)^2 = -\frac{1}{2}\left[F_{\mu\nu}F^{\mu\nu} + 2F_{\mu\nu}(*F)^{\mu\nu}\right].$$

This now precisely matches the expansion of the gauge kinetic term $\int d^2\theta W_\alpha W^\alpha$ that we encountered when discussing vector superfields.

Note that this expression should also have overall factors of the charge (essentially as $D_\mu \to \partial_\mu - ieA_\mu$ etc), and the charge is the Casimir of the representation. So what we emerge with overall is

$$\ln J = -\frac{1}{16} \int d^4x d^2\theta \frac{2t_2(\Phi)}{8\pi^2} \ln(e^\alpha) W_\alpha W^\alpha. \tag{356}$$

plus the contribution from $\ln \bar{J}$. Let us consider two special cases.

First, take the case where α is purely imaginary ($\alpha = i\theta$). Then

$$\begin{aligned} \ln J + \ln \bar{J} = i\theta \int \frac{d^4x}{16\pi^2} EB \frac{-2[\cosh t(E+B) - \cosh t(E-B)]}{\sinh tE \sinh tB} \\ &= i\theta \int \frac{d^4x}{16\pi^2} - 2EB \times \frac{t^2(E+B)^2 - t^2(E}{2t^2 EB} \\ &= i\theta \int \frac{d^4x}{16\pi^2} - 4EB. \end{aligned}$$

From this we see that we can write

$$\ln J + \ln \bar{J} = i\theta \left(\frac{1}{16\pi^2} \int d^4x - 4EB \right) = i\theta \int \frac{d^4x}{16\pi^2} F_{\mu\nu} (*F)^{\mu\nu}, \tag{358}$$

as $-4EB = F_{\mu\nu}(*F)^{\mu\nu}$.

This result should not be a surprise at all, as what we have done here is simply to reproduce the chiral anomaly, which corresponds to a rephasing of chiral fermions (and thus a rephasing of the chiral superfields).

Now let us take the case where α is purely real ($\alpha = \theta$). In this case we have not a rephasing but a rescaling of the chiral superfields. We now have

$$\begin{aligned} \ln J + \ln \bar{J} &= \theta \int \frac{d^4x}{16\pi^2} EB \frac{2[2 - \cosh t(E+B) - \cosh t(E-B)]}{\sinh tE \sinh tB} \\ &= \theta \int \frac{d^4x}{16\pi^2} 2EB \frac{[-t^2(E+B)^2 - t^2(E-B)^2]}{2t^2 EB} \\ &= -\theta \int \frac{d^4x}{16\pi^2} 2(E^2+B^2) = -\theta \int \frac{d^4x}{16\pi^2} F_{\mu\nu} F^{\mu\nu}. \end{aligned} \tag{359}$$

What we now see is a correction to the *physical gauge couplings*. This correction is known as the *Konishi anomaly*. What this says is that if we have a theory with matter fields that are not canonically normalised, then the *physical* (1PI) gauge couplings receive an anomalous 1-loop contribution

$$\frac{1}{g^2_{phys}}(\mu) = \frac{1}{g^2_{holo}}(\mu) - \frac{T_a(r)}{8\pi^2} \ln Z^r, \tag{360}$$

where the Kähler potential looks like

$$K = \ldots + Z\Phi^\dagger \Phi + \ldots$$

So we start with

$$\mathcal{L} = \int d^2\theta f_a(\Phi) W_\alpha W^\alpha + \int d^4\theta \sum_i Z_r \Phi_r^\dagger e^{qV} \Phi_r, \tag{361}$$

and this turns into

$$\frac{1}{g^2_{phys}} = \text{Re} f_a(\Phi) + \frac{(\sum_r n_r T_a(r) - 3T_a(G))}{16\pi^2} \ln \left(\frac{\Lambda^2}{\mu^2} \right) - \underbrace{\sum_r \frac{T_a(r)}{8\pi^2} \ln \det Z^r}_{\text{Konishi anomaly}}. \tag{362}$$

We now extend this to look at anomalous contributions to gauge couplings coming from non canonically-normalised gauge fields.

7.2 Vector Multiplets and the NSVZ β Function

We now look at resclings associated to a vector multiplet. We first need to set up the action. For vector fields we need to gauge-fix, which we do via Faddeev-

Popov:

$$\begin{aligned} 1 &= \int \mathcal{D}g\delta(D^\mu V^g_\mu - a)\det(D^\mu D_\mu) \\ &= \int \mathcal{D}g\mathcal{D}c\mathcal{D}\bar{c}\delta(D^\mu V^g_\mu - a)e^{-\int d^4x\bar{c}(D^\mu D_\mu)c}. \end{aligned} \tag{363}$$

Now we can also insert

$$1 = \frac{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}}{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}},$$

giving overall

$$1 = \frac{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2} \int \mathcal{D}g\mathcal{D}c\mathcal{D}\bar{c}\delta(D^\mu V^g_\mu - a)e^{-\int d^4x\bar{c}(D^\mu D_\mu)c}}{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}}. \tag{364}$$

Integrating over a, we get

$$1 = \frac{\int \mathcal{D}g\mathcal{D}c\mathcal{D}\bar{c}e^{-\int d^4x \frac{\xi(D^\mu V^g_\mu)^2}{2g^2} + \bar{c}(D^\mu D_\mu)c}}{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}}. \tag{365}$$

As standard, we then have the path integral

$$Z = \frac{\int \mathcal{D}V\mathcal{D}g\mathcal{D}c\mathcal{D}\bar{c}e^{-\int d^4x \frac{\xi(D^\mu V^g_\mu)^2}{2g^2} + \bar{c}(D^\mu D_\mu)c - F_{\mu\nu}F^{\mu\nu}}}{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}}. \tag{366}$$

As the kinetic term is gauge-invariant, we can write (using $\mathcal{D}V^g = \mathcal{D}V$)

$$\begin{aligned} Z &= \frac{\int \mathcal{D}V^g\mathcal{D}g\mathcal{D}c\mathcal{D}\bar{c}e^{-\int d^4x \frac{\xi(D^\mu V^g_\mu)^2}{2g^2} + \bar{c}(D^\mu D_\mu)c - F_{\mu\nu}F^{\mu\nu}}}{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}}, \\ &= \int \mathcal{D}g \left(\frac{\int \mathcal{D}V^g\mathcal{D}c\mathcal{D}\bar{c}e^{-\int d^4x \frac{\xi(D^\mu V^g_\mu)^2}{2g^2} + \bar{c}(D^\mu D_\mu)c - F_{\mu\nu}F^{\mu\nu}}}{\int \mathcal{D}a e^{-\int d^4x \frac{\xi}{2g^2}a^2}} \right) \end{aligned} \tag{367}$$

This is the standard Faddeev-Popov trick. Now note an important feature. If we rescale the vector multiplet, then we also need to rescale the gauge-fixing function (as $D^\mu V^g_\mu = a$ is the gauge-fixing function).

Now, the quadratic gauge field operator is, in Feynman gauge ($\xi = 1$),

$$A^{a,\mu}\left[(D_\mu)^2_{ab}\delta_{\mu\nu} - i(F^c_{\rho\sigma})(M^{\rho\sigma})_{\mu\nu}(t^c_G)_{ab}\right]A^{\nu,b}.$$

This is the quadratic operator that acts on vector fields in a background gauge field (i.e. generalising D^2_μ and $\not{D}^2$ for scalar and spinor fields). It is this operator that we have to use to regularise the path integral when we rescale the vector field measure.

So for the vector superfield we then have

$$\ln J = \alpha \left(\mathrm{Tr}_v e^{t\left[(D_\mu)^2\delta_{\mu\nu} - iF_{\rho\sigma}(M^{\rho\sigma})_{\mu\nu}\right]} - \mathrm{Tr}_a e^{t(D_\mu)^2} - \mathrm{Tr}_\lambda e^{t(\not{D})^2} - \mathrm{Tr}_{\bar{\lambda}} e^{t\not{D}^2} + \mathrm{Tr}_D e^{t(D^\mu D_\mu)} \right). \tag{368}$$

Tr_a and Tr_D cancel, to give

$$\ln J = \alpha \left(\mathrm{Tr}_v e^{t\left[(D_\mu)^2 \delta_{\mu\nu} - iF_{\rho\sigma}(M^{psigma})_{\mu\nu}\right]} - \mathrm{Tr}_\lambda e^{t\not{D}^2} - \mathrm{Tr}_{\bar{\lambda}} e^{t\not{D}^2} \right). \quad (369)$$

Let's evaluate this for the simplest case, an $SU(2)$ gauge field, with a background for the T^3 generator. The T^3 generator just leads to simple charges for $T^1 \pm iT^2$, which behave effectively as $U(1)$ fields. We can focus on these particular generators to see what the modified coupling is.

We do something very similar to what we did previously for the fermionic fields. We write

$$F^3_{\mu\nu} = \begin{pmatrix} 0 & E & 0 & 0 \\ -E & 0 & 0 & 0 \\ 0 & 0 & 0 & B \\ 0 & 0 & -B & 0 \end{pmatrix}. \quad (370)$$

We can decompose the vector field into four components, writing

$$A(x) = \begin{pmatrix} A^1 \\ A^2 \\ A^3 \\ A^4 \end{pmatrix}. \quad (371)$$

Just as for the spinor fields, we can decompose

$$A_\mu(x) = A(x) \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad (372)$$

and we have

$$-i(F_{\rho\sigma})(M^{\rho\sigma})_{\mu\nu} = \begin{pmatrix} 2iE\sigma^2 & 0 \\ 0 & 2iB\sigma^2 \end{pmatrix}. \quad (373)$$

Then the vector heat kernel becomes

$$\mathrm{Tr} e^{t(D_\mu)^2} \times \mathrm{Tr} \exp \begin{bmatrix} 2itE\sigma^2 & 0 \\ 0 & 2itB\sigma^2 \end{bmatrix},$$

with the traces factorising. This gives

$$\frac{1}{16\pi^2} \int d^4x EB \frac{2(\cosh 2tE + \cosh 2tB)}{\sinh tE \sinh tB}.$$

We now combine these with the fermionic traces to give

$$\ln J = \frac{\alpha}{16\pi^2} \int d^4x 2EB \frac{[\cosh 2tE + \cosh 2tB - \cosh t(E+B) - \cosh t(E-B)]}{\sinh tE \sinh tB}. \quad (374)$$

and so

$$\ln J = \frac{\alpha}{16\pi^2} \int d^4x 2(E^2 + B^2) \longrightarrow \frac{\alpha}{16\pi^2} \int d^4x F_{\mu\nu} F^{\mu\nu}. \quad (375)$$

The 'charge' factor is $T_2(G)$ which again comes from the fact that the vector is in the adjoint of the gauge group and so the appropriate Casimir is that applying to the adjoint. So the overall result is

$$\mathcal{D}(g_c V_c) = \mathcal{D}(V_c) \exp \left[\int d^4y \int d^2\theta \frac{2T_2(G)}{8\pi^2} (\ln g_c) W^a(g_c V_c) W^a(g_c V_c) \right]. \quad (376)$$

This gives an *anomalous* contribution to the gauge coupling for rescaling the vector superfield and making it canonically normalised.

For the physical coupling, we then have

$$\frac{1}{g^2(\mu)} = \mathrm{Re}(f_a(\Phi)) + \frac{\beta_a}{16\pi^2} \ln\left(\frac{\Lambda^2}{\mu^2}\right) + \frac{T(G)}{8\pi^2} \ln g_a^{-2}(\Phi, \bar{\Phi}, \mu) - \sum \frac{T_a(r)}{8\pi^2} \ln \det Z^r(\Phi, \bar{\Phi}, \mu). \tag{377}$$

Eq (377) relates the physical coupling - the one that actually matters in determining the strength of scattering amplitudes - to the holomorphic coupling which is protected by non-renormalisation theorems. Anomalies are 1-loop exact, and this relationship can be integrated to give the (all-orders) NSVZ β-function,

$$\beta(g) = -\frac{1}{16\pi^2} \frac{3T_2(G) - \sum_r T_2(r)(1-\gamma_r)}{1 - \frac{T_2(G)g^2}{8\pi^2}}. \tag{378}$$

Here γ_r is the anomalous dimension of the chiral superfields Φ_r. These anomalous dimensions get corrections at all orders in the loop expansion, as these are associated to renormalisation of the kinetic term for Φ_r, which comes from the Kähler potential and so is not protected against renormalisation. Note that the gauge field does not have an equivalent term: the holomorphic gauge kinetic function is 1-loop exact, as is the anomalous rescaling, and so the corrections terminate.

This β function is famous both as an 'exact' result in supersymmetric gauge theories and as a guide to understanding the general properties of supersymmetric gauge theories. However one should use the 'exactness' with care as β-functions are not scheme independent quantities. Fixed points are scheme-independent, and so the NSVZ β-function can (for example) be used to study the phase diagram of a supersymmetric theory.

Finally, in supergravity there is an additional anomaly (the Kähler-Weyl anomaly) that comes from the rescaling of $g \to \lambda g$, where g is the space-time metric. This arises as the 'natural' metric kinetic term, in a basis where holomorphy is manifest, is given not by the Einstein-Hilbert term but instead by a kinetic term

$$\int \sqrt{g} e^{K/3} \mathcal{R}. \tag{379}$$

Converting this to canonical Einstein gravity requires a rescaling of the metric, in the same way that converting the holomorphic gauge kinetic term $\int f_a(\Phi) W_\alpha W^\alpha$ to a canonically normalised kinetic term $\int W_\alpha W^\alpha$ required a rescaling of the vector superfields.

Taking this metric rescaling into account, we obtain the supergravity generalisation of the NSVZ β function, given by the Kaplunovsky-Louis formula

$$\frac{1}{g^2(\mu)} = \mathrm{Re}(f_a(\Phi)) + \frac{\beta_a}{16\pi^2} \ln\left(\frac{M_P^2}{\mu^2}\right) + \frac{T(G)}{8\pi^2} \ln g^{-2}(\Phi, \bar{\Phi}, \mu) + \frac{\left(\sum n_r T_a(r) - T(G)\right)}{16\pi^2} \hat{K}(\Phi, \bar{\Phi}) - \sum \frac{T_a(r)}{8\pi^2} \ln \det Z \tag{380}$$