DR. JOSEPH CONLON
Hilary Term 2012

SUPERSYMMETRY: EXAMPLE SHEET 1

1. Prove the following identities:

$$(\sigma^\mu)_{\alpha\dot\beta}\,(\bar\sigma_\mu)^{\dot\gamma\delta} = 2\delta^\delta_\alpha\delta^{\dot\gamma}_{\dot\beta}$$

$$\mathrm{Tr}\,(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu}$$

$$(\sigma^\mu\bar\sigma^\nu = (\sigma^\nu\bar\sigma^\mu)_\alpha{}^\beta = 2\eta^{\mu\nu}\delta_\alpha{}^\beta$$

Using these, show that

$$V^\mu = \frac{1}{2}\,(\bar\sigma^\mu)^{\dot\alpha\alpha}\,V_{\alpha\dot\alpha}$$

and the decomposition $(\frac{1}{2},0)\otimes(0,\frac{1}{2}) = (\frac{1}{2},\frac{1}{2})$:

$$\psi_\alpha\bar\chi_{\dot\alpha} = \frac{1}{2}\,(\sigma^\mu)_{\alpha\dot\alpha}\,(\psi\sigma_\mu\bar\chi)\,.$$

2. Show that $\sigma^{\mu\nu} = \frac{i}{4}\,(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu)$ satisfy the Lorentz algebra and are therefore the generators of the Lorentz group in the spinor representation.

3. Prove the explicit map from $SL(2,\mathbb{C}) \to SO(3,1)$, $\Lambda_\mu{}^\nu = \frac{1}{2}\mathrm{Tr}\left[\bar\sigma^\mu N\sigma_\nu N^\dagger\right]$.

4. Using the definition $\psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta$, prove that under $SL(2,\mathbb{C})$, ψ^α transforms as

$$\psi^\alpha \to \psi'^{,\alpha} = \psi^\beta(N^{-1})_\beta{}^\alpha.$$

Find an expression for N^{-1} in terms of N.

5. Prove that

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = -\psi_\alpha\chi^\alpha \equiv \chi\psi.$$

Check that both $\psi\chi$ and $\bar\psi\bar\chi = \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$ are scalars under $SL(2,\mathbb{C})$, and verify the Fierz identities

$$(\theta\psi)(\bar\chi\bar\eta) = -\frac{1}{2}\,(\theta\sigma^\mu\bar\eta)\,(\bar\chi\bar\sigma_\mu\psi)$$

$$(\theta\sigma_\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)$$

6. Using the fact that

$$(\sigma^{\mu\nu})_\alpha{}^\beta(\sigma_{\mu\nu})_\gamma{}^\delta = \epsilon_{\alpha\gamma}\epsilon^{\beta\delta} + \delta_\alpha{}^\delta\delta_\gamma{}^\beta,$$

prove that $(\frac{1}{2},0)\otimes(\frac{1}{2},0) = (0,0)\oplus(1,0)$, i.e.

$$\psi_\alpha\chi_\beta = \frac{1}{2}\epsilon_{\alpha\beta}(\psi\chi) + \frac{1}{2}\left(\sigma^{\mu\nu}\epsilon^T\right)_{\alpha\beta}(\psi\sigma_{\mu\nu}\chi).$$

7. Prove that the Pauli-Ljubanski vector $W_\mu \equiv \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}P^\nu M^{\rho\sigma}$ satisfies the following commutation relations:

$$[W_\mu, P_\nu] = 0, \qquad [W_\mu, W_\nu] = i\epsilon_{\mu\nu\rho\sigma}W^\rho P^\sigma,$$

$$[W^\mu, Q_\alpha] = -i(\sigma^{\mu\nu})_\alpha{}^\beta Q_\beta P_\nu.$$

Show that $W^\mu W_\mu$ is a Casimir operator of the Poincare algebra, but not of the super-Poincare algebra.

8. Write down the $N = 1$ supersymmetry algebra in four-component Dirac formalism. Assume the supersymmetry generators are Majorana spinors.

9. Write explicitly all components of the N=1 supersymmetry multiplets corresponding to:

1. Massive particles of maximum spins 1/2 and 1.
2. Massless particles of maximum helicity 2 and 1.

Verify in each case that the number of bosons equals the numbers of fermions. Explain how the Higgs mechanism works in supersymmetry (i.e. which massless multiplets turn into which massive multiplets?)