DR. JOSEPH CONLON
Hilary Term 2014

CLASSICAL MECHANICS: EXAMPLE SHEET 4

1. Write down the Hamilton-Jacobi equation for the 1-dimensional simple harmonic oscillator for a particle with unit mass, and solve for W using the substitution $W = W_0(q,\alpha) - \alpha t$ with constant α. You can leave the expression you obtain for W_0 as an integral over q.

Thereby show that the solution to the physical problem is

$$q(t) = \frac{\sqrt{2\alpha}}{\omega} \sin \omega \left(t + \beta\right),$$

where β is a constant.

Show that α is the total energy E.

2. Consider the 3-dimensional Hamiltonian for a particle in a gravitational field,

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2m} + mgz.$$

Construct the Hamilton-Jacobi equation, and solve it using separation of variables.

Using this solution, solve for the evolution of the particle and find $x(t)$, $y(t)$, $z(t)$.

3. A particle of unit mass moves in a potential of the form

$$V(q) = U \tan^2 (aq),$$

where U and a are positive constants. Find the turning points of the motion. Prove that the action variable I obeys the relation

$$\frac{aI}{\sqrt{2}} = \sqrt{E+U} - \sqrt{U},$$

where E is the total energy, and thus prove that the frequency ω depends on energy as

$$\frac{\omega}{a\sqrt{2}} = \sqrt{E+U}.$$

You may find the substitution $x = \tan(aq)$ useful for doing the action integral.

4. A particle of mass m is constrained to move under the action of gravity in the vertical (x,z) plane on a smooth cycloid curve given parametrically by

$$\begin{aligned} x &= l\left(\theta + \sin\theta\right), \\ z &= l(1-\cos\theta). \end{aligned}$$

Show that a suitable Hamiltonian is

$$H = \frac{p_\theta^2}{4ml^2(1+\cos\theta)} + mgl(1-\cos\theta).$$

Use action/angle variables to show that the frequency of oscillation of the particle is independent of its amplitude, i.e. it is the same for all initial conditions with $|\theta| < \pi$.

You may find the substitution $s = \sin(\theta/2)$ useful.

5. In this exercise we re-express the propagator for a quantum-mechanical particle as a path integral. The quantity we wish to compute is

$$U(x_f, t_f, x_i, t_i) = \langle x_f | \exp(-i\frac{H}{\hbar}(t_f - t_i))|x_i\rangle.$$

Evaluate this explicitly for $H = \frac{P^2}{2m} + V(x)$ by inserting a complete set of momentum states, $1 = \int dk|k\rangle\langle k|$.

Now, decompose the time difference $(t_f - t_i)$ into $N \gg 1$ infinitesimal time steps, and split the overall operator into $\exp(-i\frac{H}{\hbar}(t_f - t_i))$ into N parts, one for each for infinitesimal time step.

If $\epsilon \ll 1$, explain why we can write

$$\exp\left(-i\frac{\epsilon}{\hbar}\left(\frac{P^2}{2m} + V(x)\right)\right)$$

as

$$\exp\left(-i\frac{\epsilon}{\hbar}\frac{P^2}{2m}\right)\exp\left(-i\frac{\epsilon}{\hbar}V(x)\right).$$

Explain why this step requires $\epsilon \ll 1$.

For each infinitesimal time step, insert a factor of $1 = \int dx_k |x_k\rangle\langle x_k|$. This makes explicit the interpretation of the amplitude as the sum over amplitudes for all possible paths.

Evaluate this expression, by also inserting complete sets of momentum operators, $1 = \int dk|k\rangle\langle k|$. Perform the integrals over k explicitly. You should find

$$U(x_f, t_f, x_i, t_i) = \lim_{N\to\infty} \int \prod_{k=1}^{N-1} \int dx_k \left(\frac{m}{2\pi\epsilon\hbar}\right)^{N/2} \exp\left[-\frac{i\epsilon}{\hbar}\sum_{k=1}^{N}\left(\frac{m}{2}\frac{(x_k - x_{k-1})^2}{\epsilon^2} + V(x_{k-1})\right)\right).$$

By writing

$$\int_{x_i,t_i}^{x_f,t_f} \equiv \lim_{N\to\infty}\prod_{k=1}^{N-1}\int dx_k \left(\frac{m}{2\pi\epsilon\hbar}\right)^{N/2}$$

and

$$\lim_{\epsilon\to 0, N\to\infty} \epsilon \sum_{k=1}^{N} \quad\to\quad \int_{t_i}^{t_f} dt,$$

$$\lim_{\epsilon\to 0}\frac{x_k - x_{k-1}}{\epsilon} \quad\to\quad \frac{dx}{dt},$$

obtain an integral representation of the path integral.