Classical Mechanics

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Abstract

These are the current notes for the S7 Classical Mechanics course as of 9th April 2014. They contain the complete text and diagrams of the notes. There will be no more substantive revisions of these notes until Hilary 2014-15 (small typographical revisions may occur inbetween).

Contents

Chapter 1

Course Summary

This course is the S7 Classical Mechanics short option (for physicists) and also the B7 Classical Mechanics option for those doing Physics and Philosophy.

It consists of 16 lectures in total, and aims to cover advanced classical mechanics, and in particular the theoretical aspects of Lagrangian and Hamiltonian mechanics. Approximately, the first 12 lectures cover material that is examinable for both courses, whereas the last four lectures (approximately) cove material that is examinable only for B7.

General Comments

Why be interested in classical mechanics and why be interested in this course? Classical mechanics has a beautiful theoretical structure which is obscured simply by a presentation of Newton's laws along the lines of $\mathbf{F} = \mathbf{ma}$. One of the aims of this course is to reveal this structure.

Another good reason for taking this course is to understand quantum mechanics better.

> 'What do they know of quantum mechanics who only quantum mechanics know?'

The Lagrangian and Hamiltonian formulations of classical mechanics provide the classical structures that map across to the different formulations of quantum mechanics. Understanding advanced classical mechanics therefore allows you to understand quantum mechanics better, and to see how it differs and how it is similar to classicla mechanics.

In particular, Lagrangian mechanics and the action principle translate naturally to the Feynman path integral formulation of quantum mechanics, and Hamiltonian mechanics turns into the 'canonical' treatment of quantum mechanics, as for example taught in the second year quantum mechanics course.

Books

Classical mechanics is an old subject and there are many books on the topic, with a range of styles and quality. Here are some possibilities

1. Hand + Finch (Analytical Mechanics) - this has plenty of details and examples, while not being short on the number of words.

2. Kibble + Berkshire (Classical Mechanics) - a decent book which is maybe slightly lower than the level of the course, as it takes a while to get to Lagrangians

3. Landau + Liftshitz - vol I (Mechanics) - everyone should be exposed to Landau and Liftshitz at some point. It is quite terse, and with no verbosity, but a classic text.

4. Arnold - Mathematical Methods of Classical Mechanics - A mathemtically sophisticated approach to mechanics, above the level of the course.

5. Goldstein - a standard text for American graduate courses.

It is also worth consulting the lecture notes by the previous lecturers, John Magorrian and James Binney, for a similar if different take on the precise material within the course.

Errata

Please send corrections to j.conlon1@physics.ox.ac.uk. I thank Guillermo Valle for corrections to an earlier version of these notes.

Chapter 2

Calculus of Variations

A *function* takes a number as input and gives (usually) a number as output.

A *functional* takes a function as input and gives (usually) a number as output.

We start with a well-defined mathematical problem. Suppose we have a set of n coordinates, $q_1, q_2, \ldots q_n$, which are functions of time t. At any time there are also the first derivatives of these coordinates, $\dot{q}_1, \dot{q}_2, \ldots \dot{q}_n$. In classical mechanics the state of the system is set by the position and velocities: we need to know both to be able to predict the future evolution.

Now suppose we have a function (the notation $\mathcal{L}$ anticipates this being the Lagrangian)

$$\mathcal{L}(q_i, \dot{q}_i, t). \tag{2.1}$$

Given this function, we can define the functional $S[f]$ through

$$S = \int_A^B L(q_i, \dot{q}, t) dt \tag{2.2}$$

S is called the *action*. Here A and B refer to initial and final conditions: A corresponds to $t = t_0$, $q_i = q_i(t_0)$ and B refers to $t = t_f$, $q_i = q_i(t_f)$.

We require that the path $q_i(t)$ between A and B be an *extremum* of the action. This means that under a change in path of first order in smallness, $q_i(t) \to q_i(t) + \delta q_i(t)$, the first order change in the action δS vanishes. What equations does this generate for $q_i(t)$?

Suppose we vary the path slightly:

$$
\begin{aligned}
S + \delta S &= \int_A^B \mathcal{L}(q_i(t) + \delta q_i(t), \dot{q}_i(t) + \delta \dot{q}_i(t), t) \\
&= \int_A^B \mathcal{L}(q_i(t), \dot{q}_i(t), t) + \sum_i \delta q_i(t) \frac{\partial \mathcal{L}}{\partial q_i} + \sum_i \delta \dot{q}_i(t) \frac{\partial \mathcal{L}}{\partial \dot{q}_i} dt. \tag{2.3}
\end{aligned}
$$

Now, we can write

$$\sum_i \delta \dot{q}_i(t) \frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt} \left(\sum_i \delta q_i(t) \frac{\partial \mathcal{L}}{\partial q_i} \right) - \sum_i \delta q_i(t) \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial q_i} \right).$$

This enables us to write

$$S + \delta S = \int_A^B \mathcal{L}(q_i(t), \dot{q}_i(t), t) + \sum_i \delta q_i(t) \left(\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right) \right) + \left[\delta q_i(t) \frac{\partial \mathcal{L}}{\partial \dot{q}_i} \right]_A^B. \tag{2.4}$$

However, by definition $\delta q_i(t) = 0$ at A and B, and so we have

$$\delta S = \int_A^B \sum_i \delta q_i(t) \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right) dt. \tag{2.5}$$

This vanishes for arbitrary $\delta q_i(t)$ if and only if

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.6}$$

Why? If

$$\int_{x_0}^{x_1} f(x) g(x) dx = 0 \tag{2.7}$$

for arbitrary $f(x)$, then $g(x) = 0$.

To see this, prove it by contradiction. Suppose $g(x) \neq 0$ at some point x'. Then by making $f(x)$ approach the δ-function $\delta(x - x')$, you should be able to see that we can make the integral continuously approach

$$\int_{x_0}^{x_1} \delta(x - x') g(x) dx = g(x') \neq 0. \tag{2.8}$$

Therefore if $g(x)$ is ever non-zero, we can find a form of $f(x)$ that leads to a non-zero result for the integral. This establishes the contradiction and explains the result.

It follows that paths $q_i(t)$ extremising S are those satisfying

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} = 0. \tag{2.9}$$

Note that S is a *geometric* quantity: to specify it, you specify a path in coordinate space. Given this path, S returns a number.

One source of confusion in the calculus of variations if how the Lagrangian can be an independent function of q and $\dot{q}$; surely once you specify the path, you specify both q and $\dot{q}$, and so these are not independent? To resolve this confusion, it is useful to think carefully about the difference between the *action* and the *Lagrangian*. The action S is indeed a functional: it takes as arguments paths $q(t)$, and returns a number. It would not make sense to talk about the action between an independent function of $q(t)$ and $\dot{q}(t)$. You specify the path, which encompasses both positions and velocities, and feed this to the action. The Lagrangian however is a regular function, that takes as input numbers and return numbers. This function can perfecly sensibly take positions and velocities as independent quantities. For example, in classical mechanics, to know the energy of a particle, you need to know both its position and its velocity: you have to specify both, and knowing the position does not tell you the velocity (and vice-versa).

We can extend these ideas to extremisation subject to constraints.

Suppose we want to extremise the function

$$S = \int_A^B L(q_i, \dot{q}_i, t) dt \tag{2.10}$$

subject to a constraint $G(q_i, \dot{q}_i) = 0$. (We can also generalise this to the constraint $G(q_i, \dot{q}_i, t) = 0$). To do so, we use a *Lagrange multiplier*. Consider the integral

$$S' = \int_A^B \underbrace{L(q_i, \dot{q}_i, t) + \lambda G(q_i, \dot{q}_i)}_{L'}\, dt \tag{2.11}$$

now considered as a function of $(q_i, \dot{q}_i, \lambda, \dot{\lambda})$. As *lambda* does not enter the integrand explicitly, variation with respect to λ is straightforward, and we have

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}}\right) = \frac{\partial L}{\partial \dot{\lambda}} \tag{2.12}$$

turning into

$$0 = G(q_i, \dot{q}_i). \tag{2.13}$$

The Euler-Lagrange equations for L' therefore give rise to the constrain equation $G(q_i, \dot{q}_i) = 0$. The variation equations for L' then turn into

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_i}\right) - \frac{\partial L'}{\partial q_i} = 0, \tag{2.14}$$

$$G(q_i, \dot{q}_i) = 0. \tag{2.15}$$

If $G(q_i, \dot{q}_i) = G(q_i)$ (so there is no $\dot{q}$ dependence), then we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \lambda\frac{\partial G}{\partial q_i}, \tag{2.16}$$

$$G(q_i) = 0. \tag{2.17}$$

Variation subject to constraints is performed by solving these equations.

Let us now do some examples.

Example: Find the shortest distance between the points (x_0, y_0, z_0) and (x_1, y_1, z_1).

This is an example where we know the answer (a straight line) by other means. To do this by the calculus of variations, we note that the distance between two points is the path length

$$\int ds \tag{2.18}$$

where

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = ds\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}, \tag{2.19}$$

where for the purpose of this example $\dot{x} = \frac{dx}{ds}$. We therefore need to extremise

$$L = \int_A^B ds\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \tag{2.20}$$

As L has no explicit dependence on x, the Euler-Lagrange equation for the x coordinate turns into

$$\frac{\partial L}{\partial \dot{x}} = \text{constant} \tag{2.21}$$

and likewise we also obtain

$$\frac{\partial L}{\partial \dot{y}} = \text{constant}, \tag{2.22}$$

$$\frac{\partial L}{\partial \dot{z}} = \text{constant}. \tag{2.23}$$

These equations turn into

$$\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \text{constant}, \tag{2.24}$$

$$\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \text{constant}, \tag{2.25}$$

$$\frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = \text{constant}. \tag{2.26}$$

Taking ratios, we see that $\dot{x} : \dot{y} : \dot{z}$ is a constant ratio, and therefore the gradient is constant and the path describes a straight line.

Note that it does *not* follow that $\dot{x}$ is a constant, only that the ratio $\dot{x} : \dot{y} : \dot{z}$ is constant along the path. The reason is that s is simply a parameter along the path. Although the linear parametrisation is the simplest, it is not necessary and other parametrisations of the straight line are equally good. The extremisation equations cannot distinguish between these different parametrisations: this is why the Euler-Lagrange equations tell us that $\dot{x} : \dot{y} : \dot{z}$ is constant and not that $\dot{x}$ is constant.

Example 2: Find the shortest distance between (x_0, y_0) and (x_1, y_1), subject to the constraint $x^2 + y^2 = R^2$.

The relevant extremisation integral is now

$$\int_A^B ds \left(\sqrt{\dot{x}^2 + \dot{y}^2} + \lambda \left(\underbrace{x^2 + y^2 - R^2}_{constraint} \right) \right) \tag{2.27}$$

This generates the Euler-Lagrange equations as

$$\frac{d}{ds} \left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 2\lambda x, \tag{2.28}$$

$$\frac{d}{ds} \left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \right) = 2\lambda y, \tag{2.29}$$

$$x^2 + y^2 = R^2, \tag{2.30}$$

It is easiest to solve these by making the substitution $x = R\cos\theta, y = R\sin\theta$. Then $\dot{x} = -R\sin\theta\dot{\theta}$ and $\dot{y} = R\cos\theta\dot{\theta}$, giving

$$\frac{d}{ds}\left(\frac{-R\sin\theta\dot{\theta}}{\sqrt{R^2\dot{\theta}^2}}\right) = 2\lambda R\cos\theta, \tag{2.31}$$

$$\frac{d}{ds}\left(\frac{R\cos\theta\dot{\theta}}{\sqrt{R^2\dot{\theta}^2}}\right) = 2\lambda R\sin\theta, \tag{2.32}$$

giving

$$-\cos\theta\dot{\theta} = 2\lambda\cos\theta, \tag{2.33}$$
$$-\sin\theta\dot{\theta} = 2\lambda\sin\theta. \tag{2.34}$$

and so

$$\dot{\theta} = -2\lambda. \tag{2.35}$$

Note that the above holds for the case $\dot{\theta} > 0$, as we have taken $\sqrt{R^2\dot{\theta}^2} = R\dot{\theta}$. If $\dot{\theta} < 0$, then the sign of the final equation is changed.

Note that this solution includes both the shortest path, and also paths that go the other way around the circle (change the sign of λ). It also includes paths that multi-wrap the circle, and go round several times. This is in fact as must be the case, because we solve for *extremal* path lengths, and all of these paths do count as extremal path lengths.
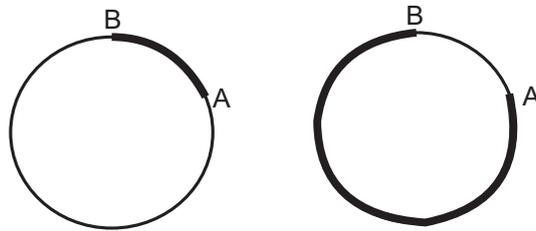

Figure 2.1: Different extremal paths around a circle.

Chapter 3

Lagrangian Mechanics

3.1 Lagrange's Equations

Let us make the transition from Newtonian mechanics, and Newtonian ways of thinking about mechanical system, to Lagrangian mechanics and Lagrange's equations. We start with a Newtonian mechanical system, where we have a system of N particles with coordinates (x_i, y_i, z_i), moving under an interaction potential $U(x_i, y_i, z_i)$. The equations of motion for this system are

$$m\ddot{x}_i = -\frac{\partial U}{\partial x_i}, \tag{3.1}$$

$$m\ddot{y}_i = -\frac{\partial U}{\partial y_i}, \tag{3.2}$$

$$m\ddot{z}_i = -\frac{\partial U}{\partial z_i}, \tag{3.3}$$

where $U = U(x_i, y_i, z_i)$ is the potential (for explicitness, this is a function of all N sets of coordinates).

Consider the function

$$L = T - U = \sum_i \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2 \right) - U(x_i, y_i, z_i). \tag{3.4}$$

If we apply the Euler-Lagrange equations to extremise the quantity

$$S = \int_{t_0}^{t_1} L dt, \tag{3.5}$$

we obtain

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) = \frac{\partial L}{\partial x_i}, \tag{3.6}$$

which gives

$$\frac{d}{dt}\left(m\dot{x}_i\right) = -\frac{\partial U}{\partial x_i}. \tag{3.7}$$

These are precisely the Newtonian equations of motion we encountered in equations (3.1) to (3.3).

This tells us that we can reformulate the Newtonian mechanics problem of particles moving under a potential U as the extremisation of the *action* S:

$$S = \int_{t_0}^{t_1} L dt = \int_{t_0}^{t_1} T - U dt. \tag{3.8}$$

We also see that the Lagrangian for this system is given by the difference of the kinetic and potential energy. Note that as extremals of minus something is also an extremal of something, we could have written $L = U - T$ and obtained the same equations of motion. However by well-established convention, $L = T - U$.

The fact that we can obtain equations of motion from extremising an action, S, given as the integral of the Lagrangian with time, $S = \int L dt$ is a very general result. Its usefulness will not go away however long you study physics. However, writing the Lagrangian as $T - U$ is a specific feature of classical mechanics systems: it is not helpful for describing particles coupled to electromagnetism or for describing the dynamics of fields.

Let us also make a philosophical aside. What is the deep structure of physics? Is physics teleological or just the blind motion of particles under forces?

In early modern physics, physics was Aristotleian and avowedly teleological. The behaviour of objects was formulated in terms of final causes. Solid bodies fell to the ground because they partook of the element earth, and the end purpose of the element earth was to move towards the centre of the globe. In contrast, objects partaking of the element fire tried to rise up into the sky and separate out from the baser elements. This picture of physics then vanished and was replaced by Newtonianism: bodies move blindly under the influence of whatever forces act on them at the time. They are not striving for anything, they just do what the forces tell them.

Action principles can be viewed as merging these two descriptions. The action principle can be formulated teleologically ('the motion of bodies, always and everywhere, is with the purpose of extremising the action of the universe'.). As we have seen however, the equations it produces are completely equivalent to viewing the motion of bodies as set purely by the forces at any one instant. As they produce the same equations, neither way of viewing physics is wrong. Instead, they carry different intuitions which are useful at different times.

This (the action principle) is called *Hamilton's principle*: motions of mechanical systems obeying Newton's equations

$$\frac{d}{dt}\left(m_i\dot{\mathbf{r}}_i\right) = -\frac{\partial U}{\partial \mathbf{r_i}}, \tag{3.9}$$

conicide with extremals of the functional

$$S = \int_{t_0}^{t_1} L dt, \tag{3.10}$$

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where $L = T - U$.

An important feature of this is that the action principle is *geometric*: it is a statament about the path taken between initial and final end-points, and about how the action varies under small changes in that path.

As a description of the dynamics of the system, the action principle can therefore be used independent of the choice of coordinates used to describe the system. We need not restrict to Cartesian coordinates. Instead, we just pick a 'good' choice of coordinates $(Q_1, \ldots Q_n, \dot{Q}_1, \ldots \dot{Q}_n)$, and

1. Write the kinetic energy T in terms of $(Q_i, \dot{Q}_i)$.

2. Write the potential energy U in terms of $(Q_i, \dot{Q}_i)$.

3. Write the Lagrangian $L(Q_i, \dot{Q}_i) = T - U$.

4. Write down and solve the Euler-Lagrange equations,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) - \frac{\partial L}{\partial Q_i} = 0. \tag{3.11}$$

In this way, we avoid having to think about forces at all. In fact, the concept of 'force' is a concept that becomes more and more deprecated the more we go on in physics. Action principles, on the other hand, become more and more important. We can use action principles as a way to obtain simple descriptions of systems for which an analysis in terms of forces is difficult.

Action principles can also be elegantly adapted to fields, where the notion of a force acting on a field is not useful. For example, the action for a pure electromagnetic field is

$$S_{EM} = -\frac{1}{4}\int dt dx dy dz F_{\mu\nu}F^{\mu\nu} \equiv E^2 - B^2, \tag{3.12}$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor.

Finally, action principles survive the transition to quantum mechanics, where the concept of a force is a hangover from classical physics about as displaced as a typewriter next to an iMac. In fact, action principles underlie one of the most elegant and powerful formulations of quantum mechanics (the Feynman path integral).

Note that while many potentials are time-independent, none of what we have done above relies on this. So even if the potential is time-dependent, and the masses of the particles are time-dependent, the Newtonian equations of motion

$$\frac{d}{dt}\left(m_i(t)\dot{\mathbf{r}}_\mathbf{i}\right) = -\frac{\partial U}{\partial r_i}(\mathbf{r}_\mathbf{i}, t) \tag{3.13}$$

follow from the Lagrangian

$$L = \sum_i \frac{1}{2} m_i(t) \dot{\mathbf{r}_i}^2 - U(\mathbf{r_i}, t) \tag{3.14}$$

as

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\mathbf{r}_i}} \right) = \frac{\partial L}{\partial \mathbf{r_i}} \tag{3.15}$$

As we have said, Newton's laws apply naturally in Cartesian coordinate systems $\mathbf{r_i}, i = 1 \ldots N$. However we can use the action principle for any coordinate system

$$q_j(\mathbf{r_i}), \dot{q}_j(\mathbf{r_i}, \dot{\mathbf{r}_i}), \qquad \text{\scriptsize 1}, j = 1 \ldots 3N.$$

There $3N$ corresponds to the fact that there are x, y, z coordinates for each particle.

q_i are then called *generalised coordinates*, and the $\dot{q}_i$ are called *generalised velocities*. The 3N-dimensional space parametrised by the q_i is called *configuration space*. In analogy to $\frac{d\mathbf{p}}{dt} = \mathbf{F}$, we have

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = \frac{\partial L}{\partial q_i}. \tag{3.16}$$

We therefore call $p_i = \frac{\partial L}{\partial \dot{q}_i}$ the *generalised momentum* and $F_i = \frac{\partial L}{\partial q_i}$ the *generalised force*.

There are two important conservation laws we can now derive. First, if the Lagrangian does not depend on a coordinate, then the corresponding generalised momentum is conserved. Such a coordinate is called *cyclic* or *ignorable*. This follows easily, as if $\frac{\partial L}{\partial q_i} = 0$, then $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0$, and so $p_i = \frac{\partial L}{\partial \dot{q}_i}$ is conserved.

An appropriate choice of generalised coordinates can make such conservation laws obvious, and allows us to identify conserved quantities.

Example: motion in a spherical potential

Consider motion in a spherically symmetric potential $V = V(r)$, where $r = \sqrt{x^2 + y^2 + z^2}$. Then

$$T = \frac{1}{2} m \left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2 \right), \tag{3.17}$$

$$U = V(r). \tag{3.18}$$

Let us transform to spherical polar coordinates. In these coordinates,

$$T = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right), \tag{3.19}$$

$$U = V(r). \tag{3.20}$$

Therefore, as $L = T - U$,

$$L = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - V(r). \tag{3.21}$$

The generalised momenta are

$$p_r = m\dot{r}, \tag{3.22}$$
$$p_\theta = mr^2\dot{\theta}, \tag{3.23}$$
$$p_\phi = mr^2\sin^2\theta\dot{\phi}. \tag{3.24}$$

As ϕ does not appear explicitly in L, p_ϕ is a constant of motion and is therefore a *conserved quantity.*

In fact, as this system is spherically symmetric, we can go further. We can always orient the sphere so that the initial conditions are $\theta = \pi/2, \dot{\theta} = 0$. In this case, the equations of motion

$$\frac{d}{dt}(p_\theta) = \frac{\partial L}{\partial\theta} \tag{3.25}$$

give

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = mr^2\sin\theta\cos\theta\dot{\phi}^2, \tag{3.26}$$

and so

$$mr^2\ddot{\theta} = -2mr\dot{r}\dot{\theta} + mr^2\sin\theta\cos\theta\dot{\phi}^2 \tag{3.27}$$

Initial conditions of $\sin\theta = 0, \dot{\theta} = 0$ then imply $\ddot{\theta} = 0$. This implies that the initial conditions are then preserved, and so $\theta = \pi/2, \dot{\theta} = 0$ remains true throughout the entire motion.

This shows that the motion of a particle under a radial force can be reduced to planar motion, with $p_\phi = mr^2\dot{\phi}$ a conserved quantity - this last quantity being of course the angular momentum. This corresponds to the fact that motion under a radial potential can be reduced to motion in a plane: this should be familiar from thinking about planetary orbits, where we can reduce a 3-dimensional problem to a 2-dimensional one.

Our intuition for mechanics tells us that there is normally another constant of the motion, the total energy.

Suppose the Lagrangian does not depend explicitly on time, and consider the function

$$H = \sum_i \dot{q}\left(\frac{\partial L}{\partial\dot{q}_i}\right) - L. \tag{3.28}$$

(We will late introduce this as the Hamiltonian.)

Suppose we have some function of the generalised coordinates and velocities, $f = f(q_i, \dot{q}_i, t)$. Along a path of the system in configuration space, we have

$$\frac{df}{dt} = \sum_i \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \sum_i \frac{\partial f}{\partial\dot{q}_i}\frac{d\dot{q}_i}{dt} + \frac{\partial f}{\partial t}. \tag{3.29}$$

We therefore find that

$$\frac{d}{dt}\left[\sum_i \dot{q}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - L\right] = \sum_i \ddot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_i \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{dL}{dt} \tag{3.30}$$

$$= \sum_i \ddot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) + \sum_i \dot{q}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \sum_i \dot{q}_i \frac{\partial L}{\partial q_i} - \sum_i \ddot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i}\right)$$

$$= \sum_i \dot{q}_i \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i}\right] \tag{3.31}$$

$$= 0, \tag{3.32}$$

using the Euler-Lagrange equations. We therefore see that if the Lagrangian is independent of time, then

$$H = \sum_i \dot{q}_i \left(\frac{\partial L}{\partial \dot{q}_i}\right) - L \tag{3.33}$$

is a constant of motion.

We can also check that if $L = T - U$, with U the potential energy independent of $\dot{q}_i$, and T a homogeneous function of degree 2 in $\dot{q}_i$ ($T = \sum_{ij} c_{ij} \dot{q}_i \dot{q}_j$), then

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = \sum_i \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T. \tag{3.34}$$

As a consequence,

$$H = (2T) - (T - U) = T + U. \tag{3.35}$$

The conservation of H is therefore equivalent to the conservation of total energy.

The above results have all originated in familiar mechanical systems, but it is important to realise that they follow from more general principles.

1. The absence of a coordinate q_i from the Lagrangian implies the conservation of the corresponding momentum $\frac{\partial L}{\partial \dot{q}_i}$.

2. The independence of the Lagrangian on time t implies that the Hamiltonian is a constant of motion.

The relationship between spatiotemporal symmetries and conserved quantities is a deep one, which we shall further discuss in Noether's theorem.

This relationship is also a relationship that extends to quantum mechanics, where you should make the connection with the fact that symmetries of the Hamiltonian give rise to operators that commute with the Hamiltonian and are thus simultaneously diagonalisable.

3.2 Rotating Frames

One important and classic application of Lagrsngian mechanics is to describe the motion of a particle in a rotating frame of reference. Such a frame is not inertial, and so the coordinates used are rotating relative to those in an inertial frame.

Examples of rotating frames include

1. Coordinates fixed on a merry-go-round on a fair.

2. North-South-East-West coordinates on the earth, as the earth is rotating daily about its own axis.

Rotating frames are characterised by the existence of *fictitious forces*, of which centrifugal force is the best known example. We intend now to derive the form of, and equations for, these forces.

Let us start with the Lagrangian for particle motion in an inertial frame,

$$L = T - U = \frac{1}{2}m\mathbf{v}^2 - U. \tag{3.36}$$

Now transform to a frame with the same origin as the inertial frame, but which is rotating with respect to it with an angular velocity $\boldsymbol{\Omega}$.
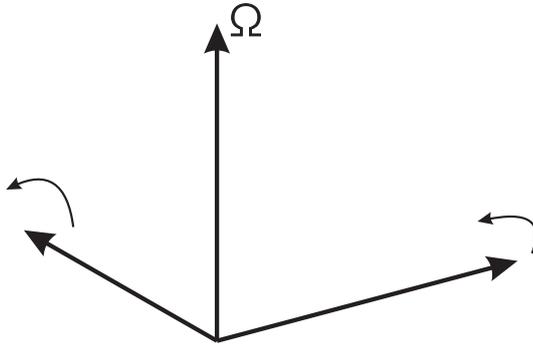

Figure 3.1: Rotating about an axis with angular velocity $\boldsymbol{\Omega}$.

The velocities in the two frames are then related by

$$\mathbf{v}_{inertial} = \mathbf{v}_{rotating} + \boldsymbol{\Omega} \times \mathbf{r}_{rotating}. \tag{3.37}$$

This is an equation which is correct, but is easy to get confused by. So let us spend a little bit of time explaining this. First, it is a *vector* equation. It means that the velocity vector in the inertial frame is related to the velocity vector in the rotating frame by an addition of a term $\boldsymbol{\Omega} \times \mathbf{r}_{rotating}$. It states the identity of the two vectors. However, what it does *not* mean is that the x-component of this vector using inertial frame x-coordinates is the same as the x-component of this

vector using rotating frame x-coordinates. That is, a vector $\mathbf{V}$ can be expanded as

$$\mathbf{V} = V_{i,r}\mathbf{e}_{i,r} = V_{i,inertial}\mathbf{e}_{i,inertial} \tag{3.38}$$

where $\mathbf{e}_{i,r}$ are basis vectors for the rotating frame and $\mathbf{e}_{i,inertial}$ are basis vectors for the inertial frames.

Returning to the main track, the significance of the action principle is that we know we can obtain the equations of motion in the rotating frame by substituting the expression for $\mathbf{v}_{inertial}$ into the expression for the Lagrangian.

We therefore have (the subscript $_r$ denotes 'rotating frame').

$$\begin{aligned}
\mathcal{L}_{rotating} &= \frac{1}{2}m(\mathbf{v}_r + \mathbf{\Omega}_r)\cdot(\mathbf{v}_r + \mathbf{\Omega}\times\mathbf{r}_r) - U \tag{3.39}\\
&= \frac{1}{2}m\mathbf{v}_r\cdot\mathbf{v}_r + m\mathbf{v}\cdot(\mathbf{\Omega}\times\mathbf{r}_r) + \frac{1}{2}m(\mathbf{\Omega}\times\mathbf{r}_r)\cdot(\mathbf{\Omega}\times\mathbf{r}_r) - U \tag{3.40}
\end{aligned}$$

We now evaluate the Euler-Lagrange equations for these coordinates

$$\begin{aligned}
dL_r &= m\mathbf{v}_r\cdot d\mathbf{v}_r + m\,d\mathbf{v}_r\cdot(\mathbf{\Omega}\times\mathbf{r}_r) + m\mathbf{v}_r\cdot(\mathbf{\Omega}\times d\mathbf{r}_r)\\
&\quad + m(\mathbf{\Omega}\times d\mathbf{r}_r)\cdot(\mathbf{\Omega}\times\mathbf{r}_r) - \frac{\partial U}{\partial \mathbf{r}_r}d\mathbf{r}_r. \tag{3.41}
\end{aligned}$$

We can use the vector triple product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$ to write this as

$$\begin{aligned}
dL_r &= m\mathbf{v}_r\cdot d\mathbf{v}_r + m\,d\mathbf{v}_r\cdot(\mathbf{\Omega}\times\mathbf{r}_r) + m\,d\mathbf{r}_r\cdot(\mathbf{v}_r\times\mathbf{\Omega})\\
&\quad + m\,d\mathbf{r}_r\cdot((\mathbf{\Omega}\times\mathbf{r}_r)\times\mathbf{\Omega}) - \frac{\partial U}{\partial \mathbf{r}_r}\cdot d\mathbf{r}_r. \tag{3.42}
\end{aligned}$$

From this we can extract the derivatives

$$\frac{\partial L}{\partial \mathbf{r}_r} = m(\mathbf{v}_r\times\mathbf{\Omega}) + m((\mathbf{\Omega}\times\mathbf{r}_r)\times\mathbf{\Omega}) - \frac{\partial U}{\partial \mathbf{r}_r}, \tag{3.43}$$

$$\frac{\partial L}{\partial \mathbf{v}_r} = m\mathbf{v}_r + m(\mathbf{\Omega}\times\mathbf{r}_r). \tag{3.44}$$

The Euler-Lagrange equations $\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}_r}\right) - \frac{\partial L}{\partial \mathbf{r}_r} = 0$ then give

$$\begin{aligned}
\overbrace{m\frac{d\mathbf{v}_r}{dt} + m\left(\dot{\mathbf{\Omega}}\times\mathbf{r}_r\right) + m(\mathbf{\Omega}\times\mathbf{v}_r)}^{\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}_r}\right)} &= \\
-m(\mathbf{v}_r\times\mathbf{\Omega}) - m((\mathbf{\Omega}\times\mathbf{r})\times\mathbf{\Omega}) + \frac{\partial U}{\partial \mathbf{r}_r} &= 0. \tag{3.45}
\end{aligned}$$

Rearranging this, we obtain

$$m\frac{d\mathbf{v}_r}{dt} = -\frac{\partial U}{\partial \mathbf{r}_r} + m\mathbf{r}_r\times\dot{\mathbf{\Omega}} + 2m\mathbf{v}_r\times\mathbf{\Omega} + m\mathbf{\Omega}\times(\mathbf{r}\times\mathbf{\Omega}). \tag{3.46}$$

In addition to the 'standard' $\frac{\partial U}{\partial \mathbf{r}_r}$ force, there are three additional 'fictitious' forces. The adjective 'fictitious' arises because these terms do not arise in inertial frames.[1]

1. $m\mathbf{r}_r \times \dot{\Omega}$: this force depends on the non-uniformity of rotation.

2. $2m\mathbf{v}_r \times \Omega$: this force is called the *Coriolis* force, and depends on the velocity of the particle.

3. $m\mathbf{\Omega} \times (\mathbf{r} \times \mathbf{\Omega})$: this is the *centrifugal* force.

What is the physical origin of these forces? The first two follow from conservation of angular momentum.

For the first case, suppose you have a particle at fixed position $\mathbf{r}_r$ in the rotating frame. In the inertial frame, this therefore has a certain amount of angular momentum as it rotates. Suppose we now increase the rotation rate of the rotating frame. If we keep the position $\mathbf{r}_r$ fixed, then the angular momentum in the inertial frame will increase, and so to conserve angular momentum the particle must experience a new 'backwards' force, which is $\mathbf{r}_r \times \dot{\Omega}$.
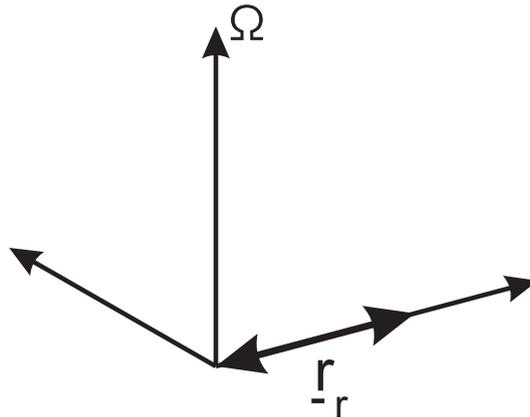

Figure 3.2: Particle at fixed position in a rotating frame.

The second case of the Coriolis force $2m\mathbf{v}_r \times \mathbf{\Omega}$ is similar. If a particle has a non-zero velocity in the rotating frame, then the effect of this is to change its angular momentum in the inertial frame. The Coriolis force acts to counter this, ensuring that angular momentum is conserved in the inertial frame.

The final case of the centrifugal force $m\mathbf{\Omega} \times (\mathbf{r} \times \mathbf{\Omega})$ is similar. By thinking in the inertial frame, it is clear that a particle with no forces on it will move at a constant velocity and thereby increase its radial separation from the origin. It therefore follows that in the rotating frame there should be a force which acts

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[1]Note that under a more advanced understanding, the gravitational force is also a 'fictitious' force, arising because we do not use a frame moving along the geodesics of spacetime.
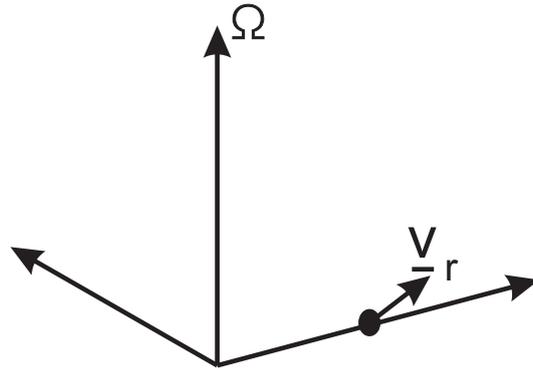
Figure 3.3: Particle with fixed velocity in a rotating frame.

to increase the radial separation from the origin. This force is the centrifugal force, with a constant tendency to expel bodies radially outwards in the plane perpendicular to the angular velocity vector.
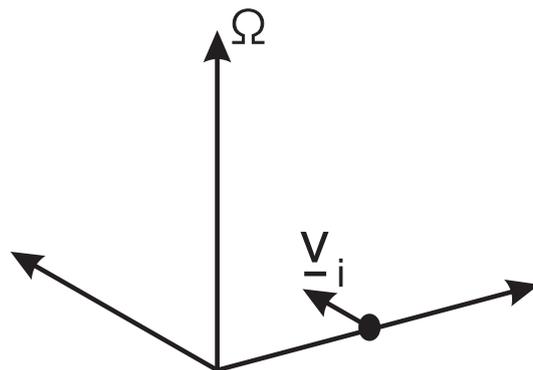

Figure 3.4: Particle with fixed velocity in an inertial frame.

Let us illustrate these fictitious forces with some examples.

1. A cylindrical glass beaker filled with liquid is rotated on a turntable at a constant angular velocity ω. Find the equation of its surface.

   We wait for the liquid to come to an equilibrium and go to coordinates that are rotating with the turntable. The relevant force is then the centrifugal force, as once everything has settled down there are no velocities within the rotating frame. In the rotating frame, the liquid is then subject to an additional centrifugal force

$$\mathbf{F} = m\omega^2\mathbf{r} \tag{3.47}$$

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directed radially outwards in cylindrical coordinates. We can regard this force as arising from a potential energy

$$U = -\frac{1}{2}m\omega^2(x^2 + y^2).$$
(3.48)

There is in addition the standard gravitational potential energy

$$V = mgz.$$
(3.49)

We obtain the equation of the surface of the liquid from the equipotential surface, which is at

$$mgz - \frac{1}{2}m\omega^2(x^2 + y^2) = mgh_0.$$
(3.50)

Here the constant h_0 is fixed as the height of the liquid at the centre of the beaker.

2. *(Foucault's pendulum)* For a plane pendulum, determine how the plane of rotation changes due to the rotation of the earth.

This is a famous experiment that can be used to demonstrate the rotation of the earth. It involves a long pendulum that makes small angle oscillations in the (x, y) plane, which are standard axes horizontal in the frame of the laboratory. Naively one would think that the oscillation direction of the pendulum would remain unchanged: if the pendulum is oscillating at an angle $\theta = \tan^{-1}(y/x)$ to the x-axis, it would continue to do so. The point however is that, due to the rotation of the earth, the (x, y)-axes are not inertial. Instead they are actually a rotating frame, inherited from the rotation of the earth.

The angular velocity vector of the earth points out of the north pole. In the above (x, y) coordinates, the z-axis points vertically out of the earth. The relevant component of the earth's angular velocity vector is the part that we resolve onto the z-axis (note that this component vanishes at the equator). This gives a Coriolis force within the (x, y) plane.

There is also a component of the earth's angular velocity vector that points in the (x, y) plane. The effect of the Coriolis force induced by this is vertical, and thus gives a tiny (and insignificant) change to the effective value of g.

The overall effect of the Coriolis effect is to modify the equations of motion to

$$\ddot{x} + \omega^2 x = 2\Omega_z \dot{y},$$
(3.51)
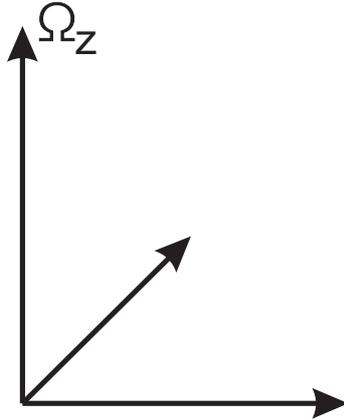$$\ddot{y} + \omega^2 y = -2\Omega_z \dot{x}.$$
(3.52)

Figure 3.5: Rotation with respect to the x-y plane.

We can solve this by writing $z = x + iy$. Multiplying equation (3.52) by i and combining, we get

$$(\ddot{x} + i\ddot{y}) + \omega^2(x + iy) = -2i\Omega_z(\dot{x} + i\dot{y}). \tag{3.53}$$

We therefore have

$$\ddot{z} + 2i\Omega_z\dot{z} + \omega^2 z = 0. \tag{3.54}$$

For the limit $\Omega_z \ll \omega$, this is solved by

$$z = \exp(-i\Omega_z t)\left[A_1 e^{i\omega t} + A_2 e^{-i\omega t}\right]. \tag{3.55}$$

The term $\exp(-i\Omega_z t)$ corresponds to a slow rotation of the plane of oscillation.

For an angle Θ from the North Pole, then $|\Omega_z| = \Omega\cos\Theta$, where $\Omega = \frac{2\pi}{24\,\text{hours}}$. This rotation of the plane of oscillation then allows a direct demonstration of the rotation of the earth.

3. *A Fake Example* A good story that you will read in several mechanics textbooks involves the 1914 Battle of the Falkland Islands. Two British battlecruisers headed down from the North Atlantic to engage a German cruiser squadron; they subsequently caught and, after an extended battle, sunk them.[2]

   According to the story, however, the British ships spent quite a while missing early on in the battle. The reason? The guns were calibrated for the Northern Hemisphere, where the Coriolis effect causes a systematic deflection in one direction. In the Southern Hemisphere, the Coriolis effect acts

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[2]The Germans had their revenge; one of the battlecruisers was later sunk at Jutland.
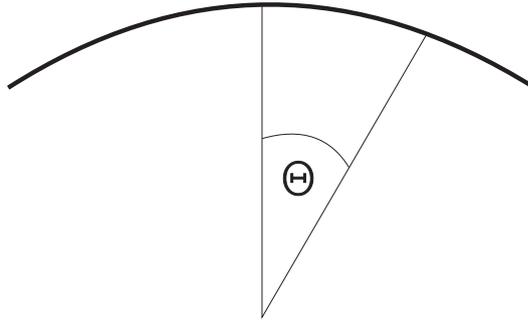
Figure 3.6: The angle with respect to the North Pole.

the other way, and so the guns were out by taking the wrong sign for the Coriolis effect.

However, while a nice story with correct physics, it is sadly an urban myth that has found its way into many textbooks.

4. *Cyclones and hurricanes*

Cyclones are caused by regions of low air pressure. This low pressure region causes air to flow in from the outer, high pressure regions. However the flow is not purely radial (as seen from above the earth), but instead modifies by the Coriolis force into an angular flow that gives the characteristic spiral shapes of hurricanes as seen by satellites.

The direction of rotation is reversed in northern and southern hemispheres. As the angular velocity vector is inherited from the earth, in the northern hemisphere the angular velocity vector points out of the surface of the earth, and in the southern hemisphere the angular velocity vector points inwards, towards the centre of the earth. This reversal of the angular velocity vector reverses the direction of the Coriolis force, and therefore cyclones rotate *anticlockwise* in the northern hemisphere and *clockwise* in the southern hemisphere.

3.3   Normal Modes

Our discussion of normal modes will be relatively brief as it is a topic that has been treated previously in the first year classical mechanics course and also in the dedicated first year Normal Modes and Waves course.

Suppose we have a mechanical system described by generalised coordinates $(q_1, \ldots, q_n)$ and a Lagrangian $L(q_i, \dot{q}_i) = T - U$.
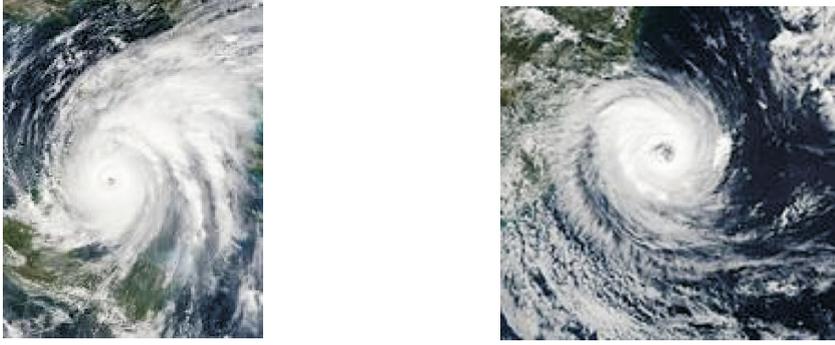
Figure 3.7: Hurricanes in the north and south Atlantic

An *equilibrium point* is a set of values $(q_1^{eq}, \ldots q_n^{eq})$ such that at the equilibrium point $(\dot{q}_1, \dot{q}_n) = 0$ solves the equations of motion. That is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{3.56}$$

is solved by $\mathbf{q} = \mathbf{q}^{eq}$ and $\dot{\mathbf{q}} = 0$.

We consider the dynamics in the vicinity of the equilibrium point. First, we can without loss of generality redefine coordinates so that our origin of generalised coordinates is at the equilibirum point. That is, we take $\mathbf{q} \to \mathbf{q} - \mathbf{q}^{eq}$, such that the equilibrium point corresponds to

$$\mathbf{q} = 0, \qquad \dot{\mathbf{q}} = 0.$$

We are also here going to restrict to regular mechanical systems, where T is a homogeneous quadratic in $\dot{\mathbf{q}}$, and so can be written

$$T = \sum a_{ij}(q)\dot{q}_i \dot{q}_j. \tag{3.57}$$

The coefficients $a_{ij}(q)$ are in general power series functions of the q_i. This allows us to perform an expansion,

$$a_{ij}(q) = a_{ij}^0 + a_{ij,k}^1 q_k + a_{ij,kl}^2 q_k q_l + \ldots \tag{3.58}$$

As we restrict to the vicinity of the equilibrium point, we can focus only on the leading (constant) term a_{ij}^0.

For a Newtonian mechanical system $T > 0$ (provided $\dot{\mathbf{q}} \neq 0$, and we can

diagonalise, rotating coordinates so that we have

$$
T = \begin{pmatrix} \dot{q}_1 & \dot{q}_2 & \cdots & \dot{q}_n \end{pmatrix} (D) \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \cdots \\ \dot{q}_n \end{pmatrix} \tag{3.59}
$$

with D a diagonal matrix. In terms of the new $\mathbf{q}'$ coordinates, we can then write

$$
T = \frac{1}{2} \sum_i \dot{q}_i'^2 \tag{3.60}
$$

$$
U = U(q'). \tag{3.61}
$$

Again, by expanding $U(q')$ about $q' = 0$, we can extract the quadratic terms as the leading non-constant contribution of the potential energy.

$$
U = U_0 + \sum_{i,j} \frac{1}{2} U_{ij} q_i' q_j'. \tag{3.62}
$$

Note that all the linear terms vanish as $q' = 0$ is an equilibrium point. The equations of motion then become

$$
\ddot{\mathbf{q}}' = \mathbf{U}\mathbf{q}'. \tag{3.63}
$$

We have neglected here any higher order terms. This neglect is a good approximation sufficiently close to the equilibrium point. What is meant by 'sufficiently' close will vary from example to example depending on the numerical factors. However the beauty of an equilibrium point is that we *can* always find values that are 'sufficiently' close.

The equations are now standard second order differential equations which we can solve using very standard techniques. We solve these equations by finding the eigenvalues and eigenvectors of $\mathbf{U}$. The eigenvectors give the *normal modes*. These are the directions in coordinate space away from the equilibrium point along which small oscillations behave as a linear simple harmonic oscillator: $\ddot{\theta} = (\pm)\lambda^2\theta$.

The signs of the eigenvalues determine whether the mode is stable or unstable. If all eigenvalues are negative ($\ddot{\theta} = -\lambda^2\theta$ in all directions), then the equilibrium point is *stable*.

Otherwise, the equilibrium point is *unstable* (as it only takes a single unstable direction to make the entire equilibrium point unstable).

Expanding general mechanical systems about their equilibrium points and determining the normal modes allows the stability of a system to be determined.

3.4 Particle in an Electromagnetic Field

One way to approach mechanics is to take the action principle as axiomatic. We write down different Lagrangians, and derive the different equations of motion that follow from these Lagrangians and compare with experiment. In this way we do not ask 'What forces act on a particle?' but instead 'What Lagrangian describes the system?'

This approach is actually the most fruitful in terms of understanding more advanced physical systems. Lagrangians remain useful long after forces disappear.

In this vein, let us consider the Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\mathbf{r} \cdot \mathbf{A}(\mathbf{r}, r) - q\phi(\mathbf{r}, t), \tag{3.64}$$

where $\mathbf{r}$ is the ordinary (x, y, z) position vector for a particle. At the level of writing this Lagrangian down, $\phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$ are just background scalar and vector functions of $\mathbf{r}$. We will subsequently interpret these as the scalar and vector potentials of electromagnetism, but to start with these are just arbitrary functions.

Note also that if we regard the Lagrangian approach as fundamental, then eq. (3.64) is something completely natural to write down. We have simply extended our previous Lagrangians, which had terms quadratic in $\dot{\mathbf{r}}$ and terms with no dependence on $\dot{\mathbf{r}}$, to include a term that is linear in $\mathbf{r}$. From this perspective, eq. (3.64) is the simplest possible way to extend the Lagrangians previously considered.

From this Lagrangian, we can then derive

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x, \tag{3.65}$$

and generally

$$\mathbf{p} = m\dot{\mathbf{r}} + q\mathbf{A}. \tag{3.66}$$

The equations of motion for this Lagrangian are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{r}}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0 \tag{3.67}$$

and so

$$\frac{d}{dt}(m\dot{x} + qA_x) = q\left(\dot{x}\partial_x A_x(\mathbf{r}, t) + \dot{y}\partial_x A_y(\mathbf{r}, t) + \dot{z}\partial_x A_z(\mathbf{r}, t) - q\partial_x\phi)(\mathbf{r}, t)\right) \tag{3.68}$$

where we have written out the equations of motion for the x-coordinate. Now,

$$\frac{dA_x(\mathbf{r}, t)}{dt} = \frac{\partial A_x(\mathbf{r}, t)}{\partial t} + \dot{x}\partial_x A_x(\mathbf{r}, t) + \dot{y}\partial_y A_x(\mathbf{r}, t) + \dot{z}\partial_z A_x(\mathbf{r}, t) \tag{3.69}$$

and so we have

$$m\ddot{x} + q\frac{\partial A_x(\mathbf{r},t)}{\partial t} + q\dot{x}\partial_x A_x(\mathbf{r},t) + q\dot{y}\partial_y A_x(\mathbf{r},t) + q\dot{z}\partial_z A_x(\mathbf{r},t)$$

$$= q\left(\dot{x}\partial_x A_x(\mathbf{r},t) + \dot{y}\partial_x A_y(\mathbf{r},t) + \dot{z}\partial_x A_z(\mathbf{r},t)\right) - q\partial_x\phi(\mathbf{r},t).$$

Cancelling terms and rearranging, this gives the equation of motions

$$m\ddot{x} = q\left(-\partial_x\phi(\mathbf{r},t) - \frac{\partial A_x(\mathbf{r},t)}{\partial t}\right)$$
$$+q\dot{y}\left(\partial_x A_y - \partial_y A_x\right) + q\dot{z}\left(\partial_x A_z - \partial_z A_x\right). \tag{3.70}$$

Now, we recall from electromagnetism that the $\mathbf{E}$ and $\mathbf{B}$ fields are related to the scalar potential $\phi(\mathbf{r},t)$ and the vector potential $\mathbf{A}(\mathbf{r},t)$ by

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \tag{3.71}$$

$$\mathbf{B} = \nabla\times\mathbf{A}. \tag{3.72}$$

and so we have

$$m\ddot{x} = qE_x + q\left(\dot{y}B_z - \dot{z}B_y\right), \tag{3.73}$$

giving

$$m\ddot{x} = q\left(E_x + (\mathbf{v}\times\mathbf{B})_x\right) \tag{3.74}$$

Generalising now to the equations of motion for all coordinates, we see that the particle Lagrangian

$$L = \frac{1}{2}m\dot{\mathbf{r}}^2 + q\mathbf{r}\cdot\mathbf{A}(\mathbf{r},r) - q\phi(\mathbf{r},t), \tag{3.75}$$

gives equations of motion

$$m\ddot{\mathbf{r}} = q\left(\mathbf{E} + (\mathbf{v}\times\mathbf{B})\right), \tag{3.76}$$

which is precisely the Lorenz force law. This Lagrangian therefore gives the equations of motion for, and this describes, the motion of a particle in an external electromagnetic field.

Let us make some general comments on this.

1. The Lagrangian cannot be written as $T - U$: the 'potential energy' term has a linear factor of $\mathbf{r}$, and so the $T - U$ structure is not valid here.

2. Note that the coupling

$$q\int dt\,\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t) - \phi(\mathbf{r},t) \tag{3.77}$$

is already relativistic. If we consider 4-vectors, we have

$$dt\,(\dot{\mathbf{r}}, 1) = dt\left(\frac{d\mathbf{r}}{dt}, \frac{dt}{dt}\right) \;\; = \;\; dt\left(\frac{d\tau}{dt}\frac{d\mathbf{r}}{d\tau}, \frac{d\tau}{dt}\frac{dt}{d\tau}\right) \tag{3.78}$$

$$= \;\; d\tau\left(\frac{d\mathbf{r}}{d\tau}, \frac{dt}{d\tau}\right)$$

$$= \;\; d\tau U^{\mu}, \tag{3.79}$$

where U^{μ} is the 4-velocity. We therefore see that we can write this coupling in the manifestly relativistic form,

$$\int dt\mathbf{r} \cdot \mathbf{A}(\mathbf{r}, t) - \phi(\mathbf{r}, t) = \int d\tau U^{\mu} A_{\mu} \tag{3.80}$$

with $A_{\mu} = (\phi, \mathbf{A})$ the 4-potential.

3. Note that this Lagrangian gives the dynamics of a particle coupled to a background electromagnetic field. It does *not* give the dynamics of the electromagnetic field itself. This is given by

$$S \;\; = \;\; \frac{1}{4}\int d^4x F_{\mu\nu}F^{\mu\nu} \tag{3.81}$$

$$= \;\; \frac{1}{2}\int d^4x\left(\mathbf{E}^2 - \mathbf{B}^2\right). \tag{3.82}$$

Here $F_{\mu nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is the electromagnetic field strength tensor.

4. The Lagrangian formulation allows us to easily work out the dynamics of a particle coupled to electromagnetism in more general coordinate systems. We simply re-express the Lagrangian in the new coordinates, and then evaluate the Euler-Lagrange equations.

3.5   Rigid Bodies

We next want to consider the dynamics of a rigid body. A *rigid body* is a system of very many particles where the distances between each pair of particles do not vary. This is an idealisation, but one that serves to capture many real systems.

A little bit of thought should be sufficient to convince oneself that rigid bodies are described by six degrees of freedom. These correspond to

- three position coordinates

- three angular coordinates: two to choose an axis, and one to rotate about that axis.

This is one of these results that is best proved by thinking about it until you convince yourself that it is correct.

For regular Newtonian mechanical systems, the dynamics of a rigid body can be described by the standard Lagrangian,

$$L = T - U. \tag{3.83}$$

In working with rigid bodies, our first goal is to find an expression for T for a rigid body that is both moving translationally and also rotating about its own axis. Now,

$$T = \sum_i \frac{1}{2} m_i \mathbf{v}^2, \tag{3.84}$$

where the sum over i is a sum over all possible constituent elements of the rigid body. Now if the body has centre of mass velocity $\mathbf{V}$, and $\mathbf{r}_i$ is the displacement vector of the element i from the centre of mass, then

$$\mathbf{v}_i = \mathbf{V}_i + \mathbf{\Omega} \times \mathbf{r}_i, \tag{3.85}$$

as per our previous calculations for the case of rotating frames. We can expand this to give

$$T = \sum_i \frac{1}{2} m_i \left(\mathbf{V} \cdot \mathbf{V} + 2\mathbf{V} \cdot (\mathbf{\Omega} \times \mathbf{r}_i) + (\mathbf{\Omega} \times \mathbf{r}_i) \cdot (\mathbf{\Omega} \times \mathbf{r}_i) \right). \tag{3.86}$$

Now,

$$\mathbf{V} \cdot (\mathbf{\Omega} \times \sum_i m_i \mathbf{r}_i) = \sum_i m_i \mathbf{r}_i \cdot (\mathbf{V} \times \mathbf{\Omega})$$

and $\sum m_i \mathbf{r}_i = 0$, as $\mathbf{r}_i$ is the displacement vector from the centre of mass, and by definition of the centre of mass $\sum m_i \mathbf{r}_i = 0$. We also have

$$
\begin{aligned}
(\mathbf{\Omega} \times \mathbf{r}_i) \cdot (\mathbf{\Omega} \times \mathbf{r}_i) &= \epsilon_{\alpha\beta\gamma} \Omega_\alpha (\mathbf{r}_i)_\beta \epsilon_{\gamma\delta\epsilon} \Omega_\delta (\mathbf{r}_i)_\epsilon \\
&= (\delta_{\alpha\delta}\delta_{\beta\epsilon} \delta_{\alpha\epsilon}\delta_{\beta\delta}) \Omega_\alpha (\mathbf{r}_i)_\beta \Omega_\delta (\mathbf{r}_i)_\epsilon \\
&= \Omega_\alpha \Omega_\delta \left(\delta_{\alpha\delta}(\mathbf{r}_i)^2 - (\mathbf{r}_i)_\alpha (\mathbf{r}_i)_\delta \right). \tag{3.87}
\end{aligned}
$$

We can therefore write

$$T = \frac{1}{2} M \mathbf{V} \cdot \mathbf{V} + \frac{1}{2} \Omega_\alpha \Omega_\delta \underbrace{\left(\sum_i m_i (\delta_{\alpha\delta}(\mathbf{r}_i)^2 - (\mathbf{r}_i)_\alpha (\mathbf{r}_i)_\delta) \right)}_{I_{\alpha\delta}}. \tag{3.88}$$

Here $I_{\alpha\delta}$ is called the *inertia tensor* and can be directly evaluated for any given shape of a rigid body.[3] We can then write

$$T = \underbrace{\frac{1}{2} M \mathbf{V} \cdot \mathbf{V}}_{\text{Linear KE of CoM}} + \underbrace{\frac{1}{2} I_{\alpha\delta} \Omega_\alpha \Omega_\delta}_{\text{Rotational KE about CoM}}, \tag{3.89}$$

---

[3]If you are frightened by the name 'tensor', you might like to call this the inertia matrix instead. It will be equivalent for all purposes here.

and also

$$L = T - U = \underbrace{\frac{1}{2}M\mathbf{V} \cdot \mathbf{V} + \frac{1}{2}I_{\alpha\delta}\Omega_\alpha\Omega_\delta}_{\text{function of generalised velocities}} - U. \tag{3.90}$$

The inertia tensor $I_{\alpha\delta}$ is a symmetric tensor. In the above equation, Ω_α gives the angular velocities about (x, y, z) coordinate axes. We can obtain the corresponding angular momentum,

$$L_\alpha = \frac{\partial L}{\partial \Omega_\alpha} = I_{\alpha\delta}\Omega_\delta. \tag{3.91}$$

You can check that this is equivalent to the more conventional definition of $\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \mathbf{v}_i$.

The relationship between L and Ω in equation (3.91) shows that the angular momentum vector L_α is *not* in general parallel to the angular velocity vector Ω_α, *except* in the case that Ω_α lies along one of the eigenvectors of the inertia tensor **I**. The eigenvectors of **I** are called the *principal axes*:

$$\mathbf{I}\boldsymbol{\Omega} = \lambda\boldsymbol{\Omega}, \tag{3.92}$$

with three eigenvalues denoted by I_1, I_2, I_3. There are three cases:

$$I_1 \neq I_2 \neq I_3 \quad : \quad \text{the asymmetric top} \tag{3.93}$$
$$I_1 = I_2 \neq I_3 \quad : \quad \text{the symmetric top} \tag{3.94}$$
$$I_1 = I_2 = I_3 \quad : \quad \text{the spherical top} \tag{3.95}$$

The equations of motion for a rigid body are then found as the Euler-Lagrange equations for

$$L = \underbrace{\frac{1}{2}M\mathbf{V} \cdot \mathbf{V} + \frac{1}{2}I_{\alpha\delta}\Omega_\alpha\Omega_\delta}_{\text{function of generalised velocities}} - U, \tag{3.96}$$

in terms of the six generalised coordinates (three translational and three rotational) that are required to specify the state of the system.

Some ways of choosing generalised coordinates are more equal than others. For problems of rotating bodies, a standard choice are the so-called *Euler angles*. However we will not go into detail here.

In certain cases of rigid bodies, there may also be constraint equations. For example, when a rigid body rolls on a rough surface, the velocity at the point of contact with the ground is required to vanish. This gives rise to *constraint equations* of the form

$$\sum_i C_{\alpha i}(q)\dot{q}_i = 0. \tag{3.97}$$

These constraints fall into two kinds.

1. Holonomic constraints, where the constrain can be integrated so as to reduce solely to a relationship between coordinates.

   An example is a cylinder rolling on a rough plane, where the translational coordinate x and the rotational coordinate θ are related by

   $$\dot{x} = R\dot{\theta}, \tag{3.98}$$

   and so we can integrate this to $x = R\theta + c$.

2. Non-holonomic constraints, which are not integrable.

   The classic example of this is a sphere rolling on a rough plane. In this case the extra degrees of freedom of the sphere means that it is not possible to integrate the constraint equation: the addition of the axis of rotation of the sphere about its pole means that we cannot use the fact that the velocity of the sphere at the point of contact with the ground vanishes to derive a relationship between coordinates.

   For non-holonomic constraints, we have to include the constraint explicitly in the extremisation procedure through use of a Lagrange multiplier.

3.6 Noether's Theorem

To conclude our discussion of Lagrangian mechanics, we now discuss Noether's theorem. This is a general theorem relating symmetries of the Lagrangian to conserved quantities.

Suppose we have a continuous set of coordinate transformations parametrised by a continuous parameter s,

$$T_s : q_i \to Q_i(s), \qquad \text{with } Q_i(0) = q_i,$$

such that

$$\mathcal{L}(q_i(t), \dot{q}_i(t), t) = \mathcal{L}(Q_i(s,t), \dot{Q}_i(s,t), t). \tag{3.99}$$

These transformation are then a map of coordinate space onto coordinate space. This map then also induces a map on velocity vectors as well.[4] The map on velocity vectors should be intuitive: as you map coordinates to coordinates, you map old trajectories to new trajectories, and thus you can evaluate the velocity vectors of the new trajectories at each point. This defines the map from old velocities to new velocities.

The information content lies in the statement that under this map the value of the Lagrangian at each point is the same, and thus we say that this map is

---

[4]In mathematics texts, you will see this stated in the form that a map on a manifold $M \to M$ induces a map on the tangent bundle, $TM \to TM$.

a *symmetry* of the Lagrangian. To keep out intuition in order, it is helpful to keep in mind that the simplest examples of such transformations are rotations or translations.

The fact that L is, by construction, invariant under these transformations tells us that

$$\frac{d}{ds}\left[L(Q_i(s,t),\dot{Q}_i(s,t),t)\right] = 0, \tag{3.100}$$

and so by the chain rule

$$\sum_i \frac{\partial L}{\partial Q_i}\frac{dQ_i}{ds} + \frac{\partial L}{\partial \dot{Q}_i}\frac{d\dot{Q}_i}{ds} = 0. \tag{3.101}$$

Now, as the coordinate mapping preserves the Lagrangian, it will also take solutions of Lagrange's equations to solutions of Lagrange's equations. (We can see this by thinking about the action principle; as the action for any path is unchanged, extremals of the action remain extremals of the action) Therefore,

$$\frac{\partial L}{\partial Q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right),$$

and so we get

$$\sum_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right)\frac{\partial Q_i}{\partial s} + \frac{\partial L}{\partial \dot{Q}_i}\frac{d}{ds}\left(\frac{\partial \dot{Q}_i}{dt}\right) = 0 \tag{3.102}$$

and therefore

$$\frac{d}{dt}\left[\sum_i \left(\frac{\partial L}{\partial \dot{Q}_i}\right)\frac{dQ_i}{ds}\right] = 0. \tag{3.103}$$

This is true for all s. However it is useful to evaluate this at the value $s = 0$ corresponding to the original coordinates. We then get

$$\frac{d}{dt}\left[\sum_i p_i \frac{dQ_i}{ds}|_{s=0}\right] = 0. \tag{3.104}$$

The reason for evaluating these at $s = 0$ is that this is the point in which coordinate space we imagine examining the evolution of the system. We therefore obtain Noether's theorem:

If a continuous coordinate mapping $q_i \to Q_i(s,t)$, $s \in \mathbb{R}$, preserves the Lagrangian:

$$L(q_i(t),\dot{q}_i(t),t) = L(Q_i(s,t),\dot{Q}_i(s,t),t)$$

then

$$\sum_i p_i \frac{dQ_i}{ds}|_{s=0}$$

is a constant of the motion.

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Noether's theorem gives the general version of the statement that linear momentum is conserved as a consequence of translational invariance, and angular momentum is conserved as a consequence of rotational invariance.

It makes the deep connection between space-time symmetries and conserved quantities: this is a connection that holds for both classical and quantum mechanics.

Chapter 4

Hamiltonian Mechanics

4.1   Hamilton's Equations

We now introduce a new approach to classical mechanics. Whereas Lagrangian mechanics tends to be more useful for solving practical calculational problems, Hamiltonian mechanics allows more of the global structure and conceptual properties of classical mechanics to be understood. It is also the approach to mechanics that transitions naturally to the canonical approach to quantum mechanics.

In Lagrangian mechanics a system is described as a motion through $(q_i, \dot{q}_i)$ space as a function of t, and the state of the system is viewed as set by the value for $(q_i, \dot{q}_i)$ at any one time.

The essence of Hamiltonian mechanics is to choose new coordinates (q_i, p_i) on this space. These coordinates are an equally good description as the original $(q_i, \dot{q}_i)$ coordinates: one can move freely back and forth between the two descriptions. The coordinates are

$$(q_i, p_i) \equiv \left(q_i, \frac{\partial L}{\partial \dot{q}_i} \right). \tag{4.1}$$

Here $p_i = p_i(q_i, \dot{q}_i)$ and $\dot{q}_i = \dot{q}_i(q_i, p_i)$.

To get the equations of motion, note that along a trajectory of the motion

$$
\begin{aligned}
dL &= \sum_i \frac{\partial L}{\partial q_i} dq_i + \sum_i \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i + \frac{\partial L}{\partial t} dt \\
&= \sum \dot{p}_i dq_i + \sum p_i d\dot{q}_i + \frac{\partial L}{\partial t} dt \\
&= \sum \dot{p}_i dq_i + d\left(\sum p_i \dot{q}_i \right) = \sum \dot{q}_i dp_i + \frac{\partial L}{\partial t}. \tag{4.2}
\end{aligned}
$$

Now we have

$$d\left(p_i \dot{q}_i - L \right) = \sum \dot{q}_i dp_i - \sum \dot{p}_i dq_i - \frac{\partial L}{\partial t}. \tag{4.3}$$

We can then define a quantity

$$H(q_i, p_i, t) = \sum p_i \dot{q}_i - L, \tag{4.4}$$

so

$$dH = \sum \dot{q}_i dp_i - \sum \dot{p}_i dq_i - \frac{\partial L}{\partial t} dt \tag{4.5}$$

along trajectories of the motion. In terms of $H(q_i, p_i)$, the equations of motion are then

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{4.6}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{4.7}$$

These equations are called *Hamilton's equations* or less commonly the *canonical equations*. Written in this form, the equations of motion are far more symmetrical than in the treatment of Lagrangian mechanics.

Let us make some comments on this.

1. Although we write $H(q_i, p_i) = \sum p_i \dot{q}_i - L$, we must read this as

$$H(q_i, p_i) = \sum p_i \dot{q}_i(q_i, p_i) - L(q_i, p_i). \tag{4.8}$$

That is, H is a function of q_i and p_i alone, and any explicit appearance of $\dot{q}_i$ must be eliminated in favour of q_i and p_i.

2. The coordinate space (q_i, p_i) described by (q, p) coordinates is often denoted as *phase space*.

3. For $\dot{q}_i = \dot{q}_i(q_i, p_i)$ to be well defined, we need $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ to be positive definite. In practice, this does not tend to be an issue.

4. We also see that

$$\left(\frac{\partial H}{\partial t}\right)_{q,p} = -\left(\frac{\partial L}{\partial t}\right)_{q,\dot{q}}. \tag{4.9}$$

Let us give some examples of Hamiltonians.

1. A particle moving in a plane described by Cartesian coordinates has

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - U(x, y). \tag{4.10}$$

and

$$p_x = m\dot{x}, p_y = m\dot{y}.$$

The Hamiltonian is then

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + U(x, y). \tag{4.11}$$

2. A particle in a plane in plane polars,

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r,\phi), \tag{4.12}$$

has

$$p_r = m\dot{r}, p_\phi = mr^2\dot{\phi}.$$

The Hamiltonian is

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + U(r,\phi). \tag{4.13}$$

3. A particle in an electromagnetic field has

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + q\left(\dot{\mathbf{x}}\cdot\mathbf{A} - \phi\right). \tag{4.14}$$

It then has

$$\mathbf{p} = m\dot{\mathbf{x}} + q\mathbf{A}. \tag{4.15}$$

The Hamiltonian is

$$
\begin{aligned}
H &= \mathbf{p}\cdot\dot{\mathbf{x}} - L \\
&= m\dot{\mathbf{x}}^2 + q\mathbf{A}\cdot\dot{\mathbf{x}} - q\dot{\mathbf{x}}\cdot\mathbf{A} + q\phi \\
&= \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi.
\end{aligned} \tag{4.16}
$$

4.2 Hamilton's Equations from the Action Principle

We here provide a brief standalone derivation of Hamilton's equations directly from the action principle. We can write the action as

$$S = \int L dt = \int \sum p_i dq_i - H dt, \tag{4.17}$$

where we have used the relationship $L = \sum p_i\dot{q}_i - H$. Instead of extremising through independent variations with respect to q and $\dot{q}$, we now extremise treating q_i and p_i as the independent variables, although for simplicity we present the argument where there is just a single conjugate (q,p) pair. Then

$$
\begin{aligned}
\delta S &= \int \delta p\, dq + p\, d(\delta q) - \left(\frac{\partial H}{\partial q}\delta q\right) dt - \left(\frac{\partial H}{\partial p}\right)\delta p\, dt \\
&= = \int \delta p\left(dq - \left(\frac{\partial H}{\partial p}\right) dt\right) + \delta q\left(-dp - \frac{\partial H}{\partial q} dt\right) + d\left(p\delta q\right). \tag{4.18}
\end{aligned}
$$

As $\int d\,[p\delta q] = 0$ as the δq variations are constrained to vanish at the endpoints, we directly recover Hamilton's equations,

$$\dot{q} = \frac{\partial H}{\partial p}, \tag{4.19}$$

$$\dot{p} = -\frac{\partial H}{\partial q}. \tag{4.20}$$

4.3 Poisson Brackets

We now want to introduce a topic that should help make the map from Hamiltonian mechanics to quantum mechanics clear. Indeed, we will make the equations of classical mechanics look almost the same as the equations of quantum mechanics!

Let $f = f(\mathbf{q}, \mathbf{p}, t)$ be some general function on phase space. Then along the trajectories of motion,

$$\begin{aligned} \frac{df}{dt} &= \sum_i \frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial f}{\partial t} \\ &= \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t}. \end{aligned} \tag{4.21}$$

We define the quantity

$$[H, f] = \sum_i \left(\frac{\partial H}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial H}{\partial p_i}\frac{\partial f}{\partial q_i} \right), \tag{4.22}$$

as the *Poisson bracket* of H and f (NB: the sign (which is entirely a matter of convention) has been reversed as to the presentation in the lectures). Generally, we define the Poisson bracket of f and g as

$$[f, g] = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right). \tag{4.23}$$

Note $[f, g] = -[g, f]$. If f is not an explicit function of t, we then have

$$\frac{df}{dt} = [f, H]. \tag{4.24}$$

and therefore f is a conserved quantity if the Poisson bracket of H and f vanishes.

This should ring bells with your knowledge of quantum mechanics, where conserved quantities in quantum mechanics are those which commute with the Hamiltonian, and also with Ehrenfest's theorem:

$$\frac{d}{dt}\langle O \rangle = \langle [O, H] \rangle + \langle \frac{\partial O}{\partial t} \rangle, \tag{4.25}$$

where these are now the commutators of quantum mechanics.

Let us enumerate some properties of Poisson brackets:

$$[f, g] = -[g, f], \tag{4.26}$$

$$[f_1 + f_2, g] = [f_1, g] + [f_2, g], \tag{4.27}$$

$$[f_1 f_2, g] = f_1 [f_2, g] + f_2 [f_1, g], \tag{4.28}$$

$$\frac{\partial}{\partial t} [f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right]. \tag{4.29}$$

Also note that

$$[q_k, f] = \frac{\partial f}{\partial p_k}, \tag{4.30}$$

$$[p_k, f] = \frac{\partial f}{\partial q_k}. \tag{4.31}$$

We therefore have

$$[q_i, q_j] = [p_i, p_j] = 0, \tag{4.32}$$

$$[q_i, p_j] = \delta_{ij}, \tag{4.33}$$

relationships which should look very familiar from quantum mechanics. It can also be shown that (the proof is not difficult, just a little fiddly)

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \tag{4.34}$$

This relationship is called *Jacobi's identity*.

If it is not already blindingly obvious, then let us state explicitly that in Poisson brackets you see the classical precursors of the commutators of quantum mechanics.

Poisson brackets play an important role in constructing conserved quantities in Hamiltonian mechanics. Suppose f and g are constants of the motion, so

$$\frac{df}{dt} = [f, H] + \frac{\partial f}{\partial t} = 0, \tag{4.35}$$

$$\frac{dg}{dt} = [g, H] + \frac{\partial g}{\partial t} = 0. \tag{4.36}$$

Then $[f, g]$ is also a conserved quantity. To prove this, note that

$$\frac{d}{dt} [f, g] = [[f, g], H] + \frac{\partial}{\partial t} [f, g]. \tag{4.37}$$

From the Jacobi identity,

$$[[f, g], H] = [f, [g, H]] + [g, [H, f]], \tag{4.38}$$

and from the general properties of Poisson brackets $\frac{\partial}{\partial t}[f, g] = \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right]$.
It then follows that

$$
\begin{aligned}
\frac{d}{dt}[f, g] &= \left[\frac{\partial f}{\partial t}, g\right] + \left[f, \frac{\partial g}{\partial t}\right] + [f, [g, H]] + [g, [H, f]] \\
&= \left[\frac{\partial f}{\partial t} - [H, f], g\right] + \left[f, \frac{\partial g}{\partial t} - [H, g]\right] \\
&= \left[\frac{df}{dt}, g\right] + \left[f, \frac{dg}{dt}\right] \tag{4.39} \\
&= 0, \tag{4.40}
\end{aligned}
$$

and so $[f, g]$ is a conserved quantity.

We can therefore use Poisson brackets to generate new conserved quantities from old: the Poisson brackets of two known conserved quantities will generate another, potentially new, conserved quantity. As there are only a finite number of conserved quantities, this of course cannot go on forever, but it may still both generate new conserved quantities and illuminate the relationship between existing ones.

As an example, let us compute the Poisson brackets of the angular momentum operators $L_i = \epsilon_{ijk} r_j p_k$. This is an exercise in being careful with summation convention and Poisson brackets. We have

$$
\begin{aligned}
[L_i, L_j] &= [\epsilon_{ilm} r_l p_m, \epsilon_{jqr} r_q p_r] \\
&= \epsilon_{ilm} \epsilon_{jqr} [r_l p_m, r_q p_r] \\
&= \epsilon_{ilm} \epsilon_{jqr} \left(p_m [r_l, r_q p_r] + r_l [p_m, r_q p_r]\right) \\
&= \epsilon_{ilm} \epsilon_{jqr} \left(p_m r_q [r_l, p_r] + p_m p_r [r_l, r_q] + r_l r_q [p_m, p_r] + r_l p_r [p_m, r_q]\right) \\
&= \epsilon_{ilm} \epsilon_{jqr} \left(p_m r_q \delta_{lr} - r_l p_r \delta_{mq}\right) \\
&= \epsilon_{ilm} \epsilon_{jql} p_m r_q - \epsilon_{ilm} \epsilon_{jmr} r_l p_r \\
&= \left(\delta_{mj}\delta_{iq} - \delta_{mq}\delta_{ji}\right) p_m r_q - \left(\delta_{ir}\delta_{lj} - \delta_{ij}\delta_{rl}\right) r_l p_r \\
&= r_i p_j - r_j p_i \\
&= \epsilon_{ijk} L_k. \tag{4.41}
\end{aligned}
$$

Comments on this

1. Remember that, despite the similiarity with quantum mechanics, here r_i and p_j are not operators: there is no ordering issue and they can be moved through each other freely.

2. We have used the identity $\epsilon_{ijk}\epsilon_{klm} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$ (which also obviously holds under cyclic permutations of the order of the indices in the ϵ tensor).

3. To check the last line, write $L_k = \epsilon_{klm} r_l p_m$ and use the identity above.

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4. This relation should look very similar to the quantum mechanical angular momentum commutation relations, e.g. $[L_x, L_y] = i\hbar L_z$.

We therefore see that the algebra of Poisson brackets is thus also very similar to the algebra of quantum mechanical operators.

4.4 Liouville's Theorem

We now want to describe another aspect of Hamiltonian mechanics, which gives the closest classical version of the uncertainty principle (lots of quantum mechanics has classical avatars - quantum mechanics was revolutionary, but not as totally revolutionary as it might first appear).

As we have said, Hamiltonian mechanics describes systems as trajectories $(q_i(t), p_i(t))$ in phase space. This we can view as a flow in phase space, analogous to the flow of a fluid. Now suppose we have a certain volume in phase space.
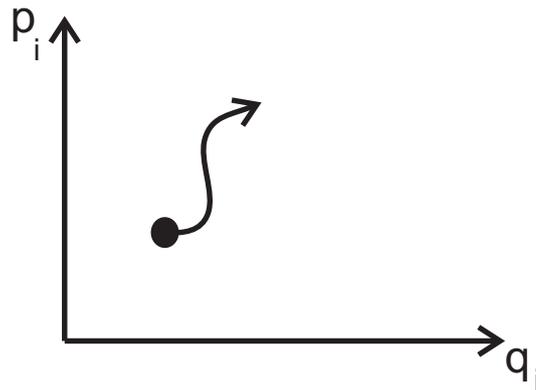
Figure 4.1: Flow in phase space

This volume we view in the same way as we view a volume of fluid that moves along with the flow of the fluid: it is enclosed by a surface, and as the surface is carried along by the flow then the volume enclosed is also carried along the flow.

We consider the flow of this volume(defined as the volume enclosed by the boundary surface as it moves along under Hamilton's equations). Liouville's theorem is the statement that

Volume in phase space is conserved during the motion of a Hamiltonian system.

To prove this, let us think about how the volume changes under a motion. We let S be the surface enclosing the volume. Near any element of hypersurface
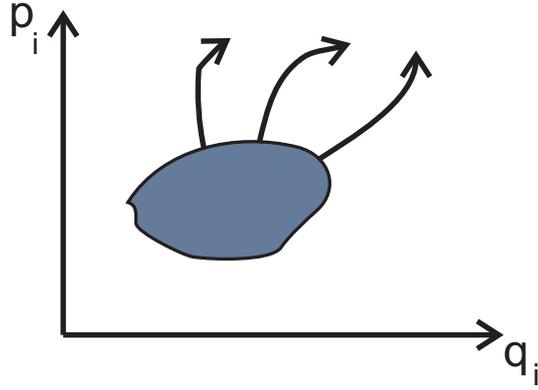
Figure 4.2: Flow of a volume in phase space

dS, with a normal vector $\hat{n}$,the instantaneous change in volume is

$$dV = \mathbf{v} \cdot \hat{n}dS \qquad (4.42)$$

where $\mathbf{v}$ is the velocity vector of the flow at that point on the surface. This is easiest to visualise for 3-dimensions, but a little thought should convince you that it holds for an arbitrary number of dimensions.

The change in the overall volume along the flow is then

$$\frac{dV}{dt} = \int_S \mathbf{v} \cdot \hat{n}dS. \qquad (4.43)$$

By Gauss's law (the divergence theorem), which holds for higher dimensions in exactly the same way as it holds for three dimensions, this can be rewritten as

$$\frac{dV}{dt} = \int_S \mathbf{v} \cdot \hat{n}dS = \int_V \nabla \cdot \mathbf{v}dV. \qquad (4.44)$$

Expressed in terms of coordinates (q_i, p_i), however,

$$
\begin{aligned}
\nabla \cdot \mathbf{v} &= \left(\frac{\partial}{\partial q_i}, \frac{\partial}{\partial p_i} \right) \cdot (\dot{q}_i, \dot{p}_i) \\
&= \frac{\partial}{\partial q_i}(\dot{q}_i) + \frac{\partial}{\partial p_i}(\dot{p}_i) \\
&= \sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i} \right) - \frac{\partial}{\partial p_i}\left(\frac{\partial H}{\partial q_i} \right) \\
&= 0. \qquad (4.45)
\end{aligned}
$$

It therefore follows that

$$\frac{dV}{dt} = 0, \qquad (4.46)$$

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and volume in phase space does not change under a Hamiltonian flow.

Why is Liouville's theorem the classical equivalent of the uncertainty principle? Let us focus on a simple mechanical system with a single pair of (Q, P) coordinates (no index).

For simplicity we are going to focus on rectangular distributions. Suppose we initially know that a system has $|Q - Q_0| < \Delta Q_0$ and $|P - P_0| < \Delta P_0$, and we then allow it to evolve under a Hamiltonian flow. As volume in phase space is conserved, there is no way it can evolve to allow us arbitrarily good knowledge of both Q and P.

In fact, the better our subsequent knowledge of Q, the worse our subsequent knowledge of P and vice-versa. If we stick for simplicity with rectangular distributions, then if at a time t_f, $|Q - Q_f| < \epsilon \Delta Q_0$, the constraint on P can only be $|P - P_f| < \frac{\Delta P_0}{\epsilon}$, as volume in phase space is conserved.
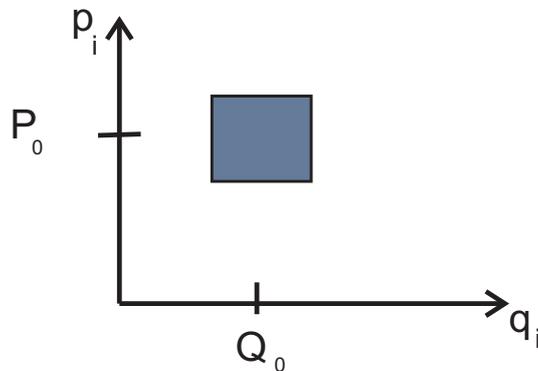
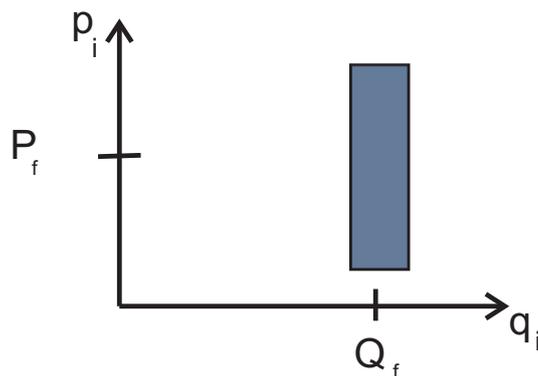Figure 4.3: Initial 'uncertainty' in phase space

Figure 4.4: Final 'uncertainty' in phase space

Note that Liouville's theorem is a however a theorem about the *total volume*

of phase space. It is not a theorem about the individual pairs of conjugate coordinates. There are (more advanced) results involving these pairs of coordinates - these results involve what are called the *Poincare integral invariants*. However that is beyond the scope of this course.

Note also that Liouville's theorem holds for any set of variables (Q_i, P_i) such that

$$\dot{Q}_i = \frac{\partial H}{\partial P_i}, \dot{Q}_i = -\frac{\partial H}{\partial Q_i}. \tag{4.47}$$

It therefore applies for all variables that are related by the *canonical transformations* we are about to discuss.

4.5 Canonical Transformations

We have seen how to get Hamilton's equations,

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \tag{4.48}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}. \tag{4.49}$$

So far, we have regarded the coordinates q_i as originating as spatial coordinates, with p_i their conjugate momenta. However we now want to think of more general coordinate transformations on phase space,

$$(q_i, p_i) \rightarrow (Q_i(q_i, p_i), P_i(q_i, p_i)), \tag{4.50}$$

which preserve Hamilton's equations: we want the transformation to be such that the equations of motion for (Q_i, P_i) are

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \tag{4.51}$$

$$\dot{P}_i = \frac{\partial H'}{\partial Q_i}. \tag{4.52}$$

for some Hamiltonian $H'(Q_i, P_i)$. This implies that (Q_i, P_i), as coordinates on phase space, retain the Hamiltonian structure. It is *not* true that general coordinate choices on phase space will keep the Hamiltonian structure - so this is restricting to special possible choices of coordinates.

Note that under such a transformation, Q_i is no longer necessarily a spatial coordinate: (Q_i, P_i) are now just some pair of coordinates on phase space that retain the Hamiltonian structure, and Q_i need no longer have an interpretation as a spatial coordinate.

Example: A very simple example of a canonical transformation is

$$Q_i = -p_i, P_i = q_i, H' = H. \tag{4.53}$$

This simply exchanges position and momentum coordinates, and it is easy to see that the new coordinates also obey Hamilton's equations.

What are the general conditions for a canonical transformation? We define a transformation to be *canonical* if

$$\left[\sum_k P_k dQ_k - H'(Q_k, P_k, t)dt\right] - \left[\sum_k p_k dq_k - H(q_k, p_k, t)dt\right] = dG, \tag{4.54}$$

where G is some function on phase space.

To see this, recall that as $H = \sum_k p_k \dot{q}_k - L$, $L = \sum p_k \dot{q}_k - H$. It therefore follows that

$$\int_{t_i}^{t_f} L dt = \int \sum_k p_k dq_k - H' dt. \tag{4.55}$$

If the above equation holds, we then have

$$\int_{(q_i,t_i)}^{(q_f,t_f)} L dt = \int_{(q_i,t_i)}^{(q_f,t_f)} \left[\sum_k P_k dQ_k - H' dt\right] - dG, \tag{4.56}$$

and so

$$G(q_f) - G(q_i) + \int_{(q_i,t_i)}^{(q_f,t_f)} L dt = \int_{(q_i,t_i)}^{(q_f,t_f)} \left[\sum_k P_k dQ_k - H' dt\right]. \tag{4.57}$$

We therefore see that extremisation of $\int L dt$ carries the same information as extremisation of $\int L' dt = \int \sum_k P_k dQ_k - H' dt$, as the G total derivative drops out. It therefore also follows that the (Q, P) coordinates obey Hamilton's equations,

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i}, \tag{4.58}$$

$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i}, \tag{4.59}$$

and so (Q_i, P_i) do indeed represent a set of canonical coordinates.

Let us make a side comment. As the extermisation of $\int L dt$ and $\int \lambda L dt$, for λ a constant, give the same result, we could have instead considered

$$\lambda \left[\sum_k P_k dQ_k - H'(Q_k, P_k, t)dt\right] - \left[\sum_k p_k dq_k - H(q_k, p_k, t)dt\right] = dG, \tag{4.60}$$

and we would also have found that (Q, P) obey Hamilton's equations. By convention, only the case with $\lambda = 1$ is called a canonical transformation ($\lambda \neq 1$ is called an *extended* canonical transformation).

We can construct canonical transformations (implicitly) through the choice of a generating function G. The different possibilities are when the function G depends on

1. Old and new coordinates $G(q, Q)$ - 'type 1 generating function'

2. Old coordinates and new momenta $G(q, P)$ - 'type 2 generating function'

3. New coordinates and old momenta $G(Q, p)$ - 'type 3 generating function'

4. Old and new momenta $G(p, P)$ - 'type 4 generating function'

We first consider the case where the generating function is $G_1(q, Q)$. It then follows that

$$dG = \frac{\partial G_1}{\partial t} dt + \frac{\partial G_1}{\partial \mathbf{q}} d\mathbf{q} + \frac{\partial G_1}{\partial \mathbf{Q}} d\mathbf{Q}. \tag{4.61}$$

We then have

$$\sum P_k dQ_k - H'(Q_k, P_k, t) dt = \left(-H(q_k, p_k, t) + \frac{\partial G_1}{\partial t} \right) dt \tag{4.62}$$
$$+ \sum_k \left(p_k + \frac{\partial G_1}{\partial q_k} \right) dq_k + \sum_k \frac{\partial G}{\partial Q_k} dQ_k.$$

As we can vary $\mathbf{q}$ and $\mathbf{Q}$ independently, we then have

$$\frac{\partial G}{\partial q_k} = -p_k, \tag{4.63}$$

$$\frac{\partial G}{\partial Q_k} = P_k, \tag{4.64}$$

$$H' = H - \frac{\partial G_1}{\partial t}. \tag{4.65}$$

These equations give, implicitly, the relationship between the coordinates $(\mathbf{q}, \mathbf{p})$ and the coordinates $(\mathbf{Q}, \mathbf{P})$.

This all sounds a bit abstract, and so let us illustrate with an example. We take a simple case: $G = \mathbf{q} \cdot \mathbf{Q}$. Then these equations give

$$\mathbf{p} = -\mathbf{Q}, \mathbf{q} = \mathbf{P}, H' = H. \tag{4.66}$$

This is the same example as we saw earlier.

We now consider the second type of canonical transformation. Here we generally write $G = \mathbf{Q} \cdot \mathbf{P} + G_2(\mathbf{q}, \mathbf{P}, t)$. (The use of the additional $\mathbf{Q} \cdot \mathbf{P}$ is to give a convenient set of equations below)

We then have

$$\sum_k P_k dQ_k - H'(Q_k, P_k, t) dt = \left(-H(q_k, p_k, t) + \frac{\partial G_2}{\partial t} \right) dt + \sum_k P_k dQ_k + Q_k dP_k$$
$$+ \sum_k \left(p_k + \frac{\partial G_2}{\partial q_k} \right) dq_k + \sum_k \frac{\partial G_2}{\partial P_k} dP_k. \tag{4.67}$$

In this case we get the implicit equations

$$\frac{\partial G_2}{\partial q_k} = -p_k, \frac{\partial G_2}{\partial P_k} = -Q_k, H = H' + \frac{\partial G_2}{\partial t}. \tag{4.68}$$

Again, we must solve these equations implicitly to obtain the coordinate transformation

Canonical transformations are general transformations on phase space. They are therefore rather far removed from the direct positional interpretation of coordinates. They are useful for revealing the general structure of mechanics, but we have now moved away from pendula hanging from rotating tops.

Note that with the canonical transformation of the second kind, it is easy for us to recover the identity transformation. If we take

$$G = \mathbf{Q} \cdot \mathbf{P} - \mathbf{q} \cdot \mathbf{P} \tag{4.69}$$

(i.e. $G_2 = -\mathbf{q} \cdot \mathbf{P}$), then we have from

$$\frac{\partial G_2}{\partial q_k} = -p_k, \frac{\partial G_2}{\partial P_k} = -Q_k, H = H' + \frac{\partial G_2}{\partial t}. \tag{4.70}$$

the relations $P_k = p_k, Q_k = q_k$. We have therefore recovered the identity transformation. We could therefore construct infinitesimal canonical transformations by starting with the above transformation, and considering infinitesimal modifications to it.

We now want to analyse the behaviour of Poisson brackets under canonical transformations. We are going to restrict here to time-independent transformations, where the function G in the definition of a canonical transformation is independent of time. In this case, we have already seen that $H' = H$.

We are going to see that Poisson brackets are invariant under canonical transformations. To do so, we first of all introduce a slightly more compact notation for Hamilton's equations. We write $\mathbf{X} = (\mathbf{q}, \mathbf{p})$. That is, $\mathbf{X}$ is a 2n-dimensional vector covering all of phase space. We also introduce the *symplectic matrix*

$$I = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \tag{4.71}$$

Using this, we can write Hamilton's equations as

$$\dot{\mathbf{X}} = I\frac{\partial H}{\partial \mathbf{X}}, \tag{4.72}$$

where we read this as a matrix equation. Using this notation, we then denote the canonical transformation $(\mathbf{q}, \mathbf{p}) \rightarrow (\mathbf{Q}, \mathbf{P})$ as $\mathbf{X} \rightarrow \mathbf{Y}$.

We can write the $\mathbf{Y}$ equations of motion using the chain rule. We have

$$
\begin{aligned}
\dot{Y}_i = \frac{\partial Y_i}{\partial X_j} \dot{X}_j &= \frac{\partial Y_i}{\partial X_j} I_{jk} \frac{\partial H}{\partial X_k} \\
&= \frac{\partial Y_i}{\partial X_j} I_{jk} \frac{\partial H}{\partial Y_l} \frac{\partial Y_l}{\partial X_k}.
\end{aligned}
\tag{4.73}
$$

We have used here the fact that X_j obeys Hamilton's equations. We can therefore write

$$
\dot{Y}_i = \left(\frac{\partial Y_i}{\partial X_j} I_{jk} \frac{\partial Y_l}{\partial X_k} \right) \frac{\partial H}{\partial Y_l}.
\tag{4.74}
$$

However, if $\mathbf{X} \to \mathbf{Y}$ is a canonical transformation, we *also* know that

$$
\dot{Y}_i = I_{il} \frac{\partial H}{\partial Y_l},
\tag{4.75}
$$

and therefore that

$$
\left(\frac{\partial Y_i}{\partial X_j} I_{jk} \frac{\partial Y_l}{\partial X_k} \right) = I_{il}.
\tag{4.76}
$$

Writing this out explicitly, we have

$$
\sum_j \frac{\partial Q_i}{\partial q_j} \frac{\partial Q_l}{\partial p_j} - \frac{\partial Q_l}{\partial q_j} \frac{\partial Q_i}{\partial p_j} = 0, \qquad i, l \in \{1, \ldots, n\},
\tag{4.77}
$$

$$
\sum_j \frac{\partial Q_i}{\partial q_j} \frac{\partial P_l}{\partial p_j} - \frac{\partial P_l}{\partial q_j} \frac{\partial Q_i}{\partial p_j} = \delta_{il}, \qquad i \in \{1, \ldots, n\}, j \in \{n+1, \ldots, 2n\}.
$$

We therefore see that

$$
[Q_i, Q_j]_{q,p} = [P_i, P_j]_{q,p} = 0.
\tag{4.78}
$$

$$
[Q_i, P_j]_{q,p} = \delta_{i,j}.
\tag{4.79}
$$

where the brackets have been evaluated with respect to (q, p) coordinates.

We therefore see that a canonical transformation preserves the fundamental Poisson brackets:

$$
[Q, P]_{Q,P} = [Q, P]_{q,p}.
\tag{4.80}
$$

We can now use this to show that canonical transformations preserve *all* Poisson brackets:

$$
[f, g]_{Q,P} = [f, g]_{q,p}
\tag{4.81}
$$

where f and g are some arbitrary functions on phase space.

To show this, we again use the notation $\mathbf{Y} = (\mathbf{Q}, \mathbf{P})$ and $\mathbf{X} = (\mathbf{q}, \mathbf{p})$. Then

$$
[f, g]_{Q,P} = \sum_{j,k} \frac{\partial f}{\partial Y_i} I_{jk} \frac{\partial g}{\partial Y_k}
$$

$$
\begin{aligned}
&= \sum_{j,k,\alpha,\beta} \frac{\partial f}{\partial X_\alpha} \frac{\partial X_\alpha}{\partial Y_j} I_{jk} \frac{\partial X_\beta}{\partial Y_k} \frac{\partial g}{\partial X_\beta} \\
&= \sum_{\alpha,\beta} \frac{\partial f}{\partial X_\alpha} \left[\sum_{j,k} \left(\frac{\partial X_\alpha}{\partial Y_j} I_{jk} \frac{\partial X_\beta}{\partial Y_k} \right) \right] \frac{\partial g}{\partial X_\beta} \\
&= \sum_{\alpha,\beta} \frac{\partial f}{\partial X_\alpha} I_{\alpha\beta} \frac{\partial g}{\partial X_\beta} \\
&= [f,g]_{q,p}. \tag{4.82}
\end{aligned}
$$

This therefore shows that the Poisson bracket is invariant under canonical transformations.

We can therefore also interpret canonical transformations as transformations that preserve the Poisson bracket structure on phase space. In some approaches, you can start here and give this as the definition of canonical transformations.

We now want want to discuss an example of a flow of canonical transformations; a continuous family of canonical transformations. This family comes from shifting coordinates along the Hamiltonian flow lines in phase space. If we evolve from time t to time $t + \tau$, then we take

$$
(q_i(t), p_i(t)) \rightarrow (q_i(t+\tau), p_i(t+\tau)), \tag{4.83}
$$

with

$$
\dot{q}_i(t+\tau) = \frac{\partial H(t+\tau)}{\partial p_i(t+\tau)}, \qquad \dot{p}_i(t+\tau) = -\frac{\partial H(t+\tau)}{\partial q_i(t+\tau)}. \tag{4.84}
$$

We can use this to define a map from $(\mathbf{q}, \mathbf{p}) \rightarrow (\mathbf{Q}, \mathbf{P})$, indeed a family of maps

$$
\begin{aligned}
Q_\tau(q_0, p_0) &= q(q_0, p_0, \tau). \tag{4.85} \\
P_\tau(q_0, p_0) &= p(q_0, p_0, \tau), \tag{4.86}
\end{aligned}
$$

where $q(q_0, p_0, \tau)$ denotes evolving (q_0, p_0) forward in time by τ, and then extracting the q coordinate.

This map just involves sliding coordinates back (or forward) along the Hamiltonian flow lines. Because of this, it is clear that the new coordinates

1. Obey Hamilton's equations (with the Hamiltonian also shifted to $H(t) \rightarrow H(t+\tau)$ if necessary).

2. And therefore preserve the Poisson bracket structure.

Therefore motion along the flow lines of the Hamiltonian generates a family of canonical transformations.

We can extend these ideas towards conserved quantities in quantum mechanics. First, note that the Hamiltonian generated a flow in phase space via

$$\dot{q}_i = [q_i, H] = \frac{\partial H}{\partial p_i}, \tag{4.87}$$

$$\dot{p}_i = [p_i, H] = -\frac{\partial H}{\partial q_i}. \tag{4.88}$$

We can extend this to generate a flow on phase space for *any* function $G(\mathbf{q}, \mathbf{p})$. We could always view $G(\mathbf{q}, \mathbf{p})$ as a 'fake' Hamiltonian, for a mechanical system on phase space.

This therefore generates a flow

$$\frac{dq_i}{d\lambda} = [q_i, G] = \frac{\partial G}{\partial p_i}, \tag{4.89}$$

$$\frac{dp_i}{d\lambda} = [p_i, G] = -\frac{\partial G}{\partial q_i}, \tag{4.90}$$

in terms of a parameter λ which takes the place of time. This flow generates a mapping $(q_i(0), p_i(0)) \rightarrow (q_i(\lambda), p_i(\lambda))$, defining the integral curves of G.

Let us give some examples.

1. $G = q_i$ generates the flow $\mathbf{q} \rightarrow \mathbf{q}, \mathbf{p} \rightarrow \mathbf{p} - \lambda$

2. $G = p_i$ generates the flow $q_i \rightarrow q_i + \lambda, p_i \rightarrow p_i$.

   (note the quantum connection between $-i\hbar\frac{\partial}{\partial x}$ and translations $x \rightarrow x + \epsilon$)

3. Consider a system with $(q_1, q_2, q_3, p_1, p_2, p_3)$ and consider $G = q_1 p_2 - q_2 p_1$. Then

$$\begin{aligned} q_1 &\rightarrow q_1 - \epsilon q_2 & p_1 &\rightarrow p_1 - \epsilon p_2, \\ q_2 &\rightarrow q_2 + \epsilon q_1 & p_2 &\rightarrow p_2 + \epsilon p_1, \\ q_3 &\rightarrow q_3 & p_3 &\rightarrow p_3. \end{aligned} \tag{4.91}$$

This is precisely recognised as a rotation about the z-axis, and the close connection of G to the quantum mechanical angular momentum operator $L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$. We can therefore see the close relationship between transformation generating operators and induced flows in phase space.

We can also show that, given a general function $F(\mathbf{q}, \mathbf{p})$, its derivative along the integral curves of G is

$$\frac{dF}{d\lambda} = [F, G]. \tag{4.92}$$

To prove that, suppose G is a Hamiltonian. Then this follows immediately from our previous results on evolution in Hamiltonian systems.

It follows from the G flow equations that an infinitesimal transformation generated by G is

$$\begin{aligned} \mathbf{q} &\rightarrow& \mathbf{q} + \epsilon \frac{\partial G}{\partial \mathbf{p}}, \\ \mathbf{p} &\rightarrow& \mathbf{p} - \epsilon \frac{\partial G}{\partial \mathbf{p}}. \end{aligned} \tag{4.93}$$

G is called a *symmetry* of the Hamiltonian if H is invariant under the map. This requires

$$\begin{aligned} 0 = \delta H &=& \frac{\partial H}{\partial \mathbf{q}} \cdot \delta \mathbf{q} + \frac{\partial H}{\partial \mathbf{p}} \cdot \delta \mathbf{p} \\ &=& \epsilon \left(\frac{\partial H}{\partial \mathbf{q}} \cdot \frac{\partial G}{\partial \mathbf{p}} - \frac{\partial H}{\partial \mathbf{p}} \frac{\partial G}{\partial \mathbf{q}} \right) \\ &=& \epsilon [H, G]. \end{aligned} \tag{4.94}$$

Therefore, a symmetry of the Hamiltonian has a vanishing Poisson bracket with it. As

$$\dot{G} = [G, H], \tag{4.95}$$

we also see that a symmetry of the Hamiltonian is conserved (cf QM).

We are familiar with the relationship between symmetries and conserved quantities from Noether's theorem.

In this case however, we see that conserved quantities also generate symmetries. Given a conserved quantity G, we can use this to generate a flow in phase space that preserves the Hamiltonian. While Noether's theorem only went in one direction, from symmetries to conserved quantities, here the result goes in two directions.

Chapter 5

Advanced Topics

This part of the notes contains the topics that are examinable for the B7 Physics and Philosophy course but are non-examinable for the S7 Classical Mechanics course.

5.1 The Hamilton-Jacobi Equation

The first part of this course made a lot of use of the *action*. We originally introduced the action as an integral along a path from $(\mathbf{q}_i, t_i)$ to $(\mathbf{q}_f, t_f)$ as

$$S = \int_{q_i}^{q_f} L \, dt. \tag{5.1}$$

In this way of thinking, the action was defined as a *functional*: $S = S[\mathbf{q}(t)]$.

We now extend this to provide a new perspective on the action. We can use the above to give us a way of thinking instead of the action as a *function*: $S = S(\mathbf{q}_i, t_i, \mathbf{q}_f, t_f)$.

What do we mean by this? Suppose we start by fixing the initial conditions $\mathbf{q}_i$ and t_i. Then, given a final time t_f and final coordinates $\mathbf{q}_f$, the least action principle determines for us the true trajectory from $(t_i, \mathbf{q}_i)$ to $(t_f, \mathbf{q}_f)$ - and therefore it also determines for us the action evaluated along the true trajectory. We can then define the action *function* $S(\mathbf{q}_i, t_i, \mathbf{q}_f, t_f)$ as the value of the action evaluated along the true path that starts at $\mathbf{q}_i$ at time t_i and ends at $\mathbf{q}_f$ at time t_f.

Although the action function is properly a function of both initial and final conditions, we shall often regard the initial conditions as fixed and focus only on the dependence on the final coordinates and time: so notation $S(\mathbf{q}, t)$ is shorthandfor $\mathbf{q} = \mathbf{q}_f$ and $t = t_f$.

Having defined the action function, let us work out its partial derivatives. We start with the spatial variation. We know from our study of Lagrangian

mechanics that the variation of action with a *path* is

$$\delta S = \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{q_i}^{q_f} + \int_{t_i}^{t_f}\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}\right)\delta q_i dt. \tag{5.2}$$

Along a true path, we know that the latter part will always vanish - as Lagrange's equations are always satisfied along a true path.

However, if we now allow $\delta \mathbf{q}(q_f)$ to vary, then δS is no longer zero. In fact we have

$$\delta S = \frac{\partial L}{\partial \dot{q}_i}\delta q_i, \tag{5.3}$$

and so we see that

$$\frac{\partial S(\mathbf{q},t)}{\partial q_i} = p_i. \tag{5.4}$$

We can also work out the variation with respect to time. To do so, we consider a true trajectory in configuration space (i.e. one satisfying the equations of motion). Then, as $S = \int L dt$, we know that

$$\frac{dS}{dt} = L, \tag{5.5}$$

where this involves the *total* derivative of the action along a true trajectory. However, we can of course also write

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \sum_i \frac{\partial S}{\partial q_i}\frac{dq_i}{dt}, \tag{5.6}$$

and so

$$\begin{aligned}\frac{\partial S}{\partial t} &= L - \sum_j p_j \dot{q}_j \\ &= -H. \tag{5.7}\end{aligned}$$

We therefore derive the partial derivatives of $S(\mathbf{q},t)$:

$$\frac{\partial S}{\partial t} = -H, \tag{5.8}$$

$$\frac{\partial S}{\partial \mathbf{q}} = \mathbf{p}. \tag{5.9}$$

Here H and p_i should be viewed as evaluated at $(\mathbf{q},t)$ for the unique true path such that $\mathbf{q}_f = \mathbf{q}$ and $t_f = t$. We can then also write

$$dS = \sum_j p_j dq_j - H dt. \tag{5.10}$$

What does this then tell us about the function $S(\mathbf{q}, t)$? Note that this is a function purely on configuration space: in contrast to the Lagrangian, it does not depend on generalised velocities at all. From the above, it also satifies the equation

$$\frac{\partial S}{\partial t} = -H(q_f, p_f) = -H(q_f, \frac{\partial S}{\partial q_f}, t). \tag{5.11}$$

We can therefore write this as

$$\frac{\partial S(\mathbf{q}, t)}{\partial t} + H(\mathbf{q}, \frac{\partial S}{\partial \mathbf{q}}, t) = 0. \tag{5.12}$$

This equation, satisfied by $S(\mathbf{q}, t)$ is called the *Hamilton-Jacobi equation.*

Let us make one comment about the Hamilton-Jacobi equation before we proceed to discuss it further. The Hamilton-Jacobi equation is not so useful for actually solving mechanical problems. If you want to understand how an ellipsoid rolls on a rotating sphere, this is not the place to start. Rather, just like Hamiltonian mechanics, its importance is in the conceptual insights it gives us. In particular it will prove fruitful in making more connections to quantum mechanics and in particular to the Schrödinger equation.

Let us set off on our path of understanding the Hamilton-Jacobi equation. Suppose we have obtained some solution $W(\mathbf{q}, t)$ to the Hamilton-Jacobi equation, so

$$\frac{\partial W}{\partial t} = -H(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}, t). \tag{5.13}$$

Given this solution, we can then construct evolution equations

$$\dot{q}_i = \frac{\partial H}{\partial p}\big|_{p_i = \frac{\partial W}{\partial q_i}}. \tag{5.14}$$

This therefore provides a set of *first order* equations of motion on configuration space. That is, given a particular solution $W(\mathbf{q}, t)$ of the Hamilton-Jacobi equations, we can use this solution to construct first order equations of motion on configuration space.

This is interesting, as we recall that Lagrangian mechanics gave rise to *second order* equations of motion on configuration space. What has happened to the extra n degrees of freedom? These have disappeared into the particular choice of solution of the Hamilton-Jacobi equation. Recall that the action function is really $S(\mathbf{q}_i, t_i, \mathbf{q}_f, t_f)$, but we then suppressed the first coordinates $(\mathbf{q}_i, t_i)$ and focused only on the dependence on $(\mathbf{q}_f, t_f)$. Solutions to the Hamilton-Jacobi equation come in continuous families, which we could parametrise by the choice of $\mathbf{q}_i$, and it is by just looking at one particular solution that we have effectively reduced the dimensionality of the system.

Another way of thinking of this is to say that the particular solution $W(\mathbf{q}, t)$ to the Hamilton-Jacobi equation that we focus on is a bit like a choice of a

wavefunction, and the Hamilton-Jacobi equation is a bit like the Schrödinger equation: by picking a particular solution, we are specifying a lot of information about the state of the system.

We can also check that the evolution equations set by eq. (5.14) also obey the other Hamilton's equation $\dot{p}_i = -\frac{\partial H}{\partial q_i}$.

We have

$$\dot{p}_i = \frac{d}{dt}\left(\frac{\partial W}{\partial q_i}\right) = \sum_j \frac{\partial^2 W}{\partial q_i \partial q_j}\frac{dq_j}{dt} + \frac{\partial^2 W}{\partial q_i \partial t}. \tag{5.15}$$

However, we know that

$$\frac{\partial W}{\partial t} = -H(\mathbf{q}, \frac{\partial W}{\partial \mathbf{q}}, t), \tag{5.16}$$

and so

$$\frac{\partial^2 W}{\partial q_i \partial t} = -\frac{\partial H}{\partial q_i} - \sum_j \frac{\partial H}{\partial p_j}\frac{\partial^2 W}{\partial q_i \partial q_j} \tag{5.17}$$

$$= -\frac{\partial H}{\partial q_i} - \sum_j \dot{q}_j \frac{\partial^2 W}{\partial q_i \partial q_j}. \tag{5.18}$$

We therefore see that $\dot{p}_i = -\frac{\partial H}{\partial q_i}$, as required.

In the case that the Hamiltonian has no explicit dependence on time, we can find a particularly simple and special solution of the Hamilton-Jacobi equation. In this case, we take

$$W(\mathbf{q}, t) = W^0(\mathbf{q}) - Et. \tag{5.19}$$

The Hamilton-Jacobi equation then becomes

$$E = H(\mathbf{q}, \frac{\partial W^0}{\partial \mathbf{q}}). \tag{5.20}$$

You should see hints in the above of the time-independent Schrödinger equation, and the way that solutions of the time-independent Schrödinger equations evolve with the simple time-dependence,

$$\psi(\mathbf{r}, t) = \psi_0(\mathbf{r})e^{-iEt}. \tag{5.21}$$

We can now make this even more precise. We consider the Schrödinger equation for a simple Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}). \tag{5.22}$$

and so

$$i\hbar\frac{\partial \psi}{\partial t} = \hat{H}\psi = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial q^2} + V(q)\psi. \tag{5.23}$$

We consider solutions of this, and decompose the wavefunction into modulus and phase:

$$\psi(q,t) = R(q,t) \exp\left(\frac{iW(q,t)}{\hbar}\right). \tag{5.24}$$

The use of W here is deliberate. Let us now plug this form for $\psi(q,t)$ into the Schrödinger equation. We have

$$i\hbar\frac{\partial\psi}{\partial t} = \left(\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial q^2} + V(q)\right)\psi \tag{5.25}$$

This becomes

$$i\hbar\left(\frac{\partial R}{\partial t} + \frac{iR}{\hbar}\frac{\partial W}{\partial t}\right) = \frac{-\hbar^2}{2m}\left(\frac{\partial^2 R}{\partial q^2} + \frac{2i}{\hbar}\frac{\partial R}{\partial q}\frac{\partial W}{\partial q} - \frac{R}{\hbar^2}\left(\frac{\partial W}{\partial q}\right)^2 + \frac{iR}{\hbar}\frac{\partial^2 W}{\partial q^2}\right) + V(q)R. \tag{5.26}$$

We now take a limit where $\hbar\left(\frac{\partial^2 W}{\partial q^2}\right) \ll |\frac{\partial W}{\partial q}|^2$. This is a limit that may be familiar to you from the A3 Further Quantum course. This approximation corresponds to the case of a potential that varies slowly with respect to the de Broglie wavelength of the particle. That is, the phase frequency of the particle is approximately constant with position. At constant potential, a particle wave behaves as $e^{i\sqrt{E-V}x}$. Provided V varies slowly, then the phase frequency is approximately constant with position, and the scale of oscillations set by the de Broglie wavelength is much smaller than the scale of variation of the potential.
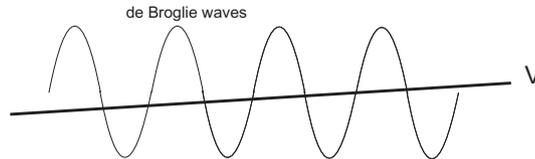

Figure 5.1: The WKB limit: de Broglie wavelength is much shorter than the scale of variation of the potential.

If we take this approximation, we have

$$i\hbar\left(\frac{\partial R}{\partial t} + \frac{iR}{\hbar}\frac{\partial W}{\partial t}\right) = \frac{-\hbar^2}{2m}\left(\frac{\partial^2 R}{\partial q^2} + \frac{2i}{\hbar}\frac{\partial R}{\partial q}\frac{\partial W}{\partial q} - \frac{R}{\hbar^2}\left(\frac{\partial W}{\partial q}\right)^2\right) + V(q)R. \tag{5.27}$$

If we now take the classical limit $\hbar \to 0$ and focus on terms of lowest ($\hbar^0$) order in $\hbar$, we have

$$-R\frac{\partial W}{\partial t} = \frac{R}{2m}\left(\frac{\partial W}{\partial q}\right)^2 + V(q)R, \tag{5.28}$$

and so

$$\frac{\partial W(\mathbf{q}, t)}{\partial t} + \frac{1}{2m} \left(\frac{\partial W}{\partial q} \right)^2 + V(q) = 0, \tag{5.29}$$

giving

$$\frac{\partial W(\mathbf{q}, t)}{\partial t} + H\left(\mathbf{q}, \mathbf{p} = \frac{\partial W}{\partial \mathbf{q}} \right) = 0. \tag{5.30}$$

We therefore see that in a semi-classical limit the *phase of the quantum wavefunction* obeys the Hamilton-Jacobi equation.

This makes the case that one should think of solutions to the Hamilton-Jacobi equation as phases. Having done this we can then make a further connection, to optics, through the *optico-mechanical analogy.*

Suppose that we have a conventional Newtonian mechanical system, so that $\mathbf{p} = m\mathbf{v}$. Then $\frac{\partial S}{\partial \mathbf{q}} = \mathbf{p}$ implies that we have

$$m\dot{\mathbf{q}} = \frac{\partial S}{\partial \mathbf{q}}, \tag{5.31}$$

and so the direction of motion is always orthogonal to the stationary surfaces of S. We can therefore view the motion in configuration space as one orthogonal to the wavefronts defined by $S[\mathbf{q}, t] = \text{constant}$.

In the case when the Hamiltonian does not depend explicitly on time, we can push this even further. Here we have

$$\begin{aligned} S[\mathbf{q}, t] &= S^0(\mathbf{q}) - Et, \\ E &= H\left(\mathbf{q}, \frac{\partial S^0}{\partial \mathbf{q}} \right). \end{aligned} \tag{5.32}$$

Then $\mathbf{p} = \frac{\partial S}{\partial \mathbf{q}} = \frac{\partial S^0}{\partial \mathbf{q}}$, and so motion for all times is determined to be orthogonal to the 'constant phase' wavefronts of $S^0 = \text{constant}$.

The magnitude of the momentum at each point is set by the requirement that $H(\mathbf{q}, \mathbf{p}) = E$ at any point.

This formulation is similar to the motion of light waves in media of varying refractive index: light moves along directions orthogonal to stationary phase. This analogue between light wave propagation and the motion of mechanical systems is the *optico-mechanical analogue.*

5.2 The Path Integral Formulation of Quantum Mechanics

We now want to describe another quantum/classical connection. This refers to the origin of Hamilton's least action principle from the path integral formulation of quantum mechanics.
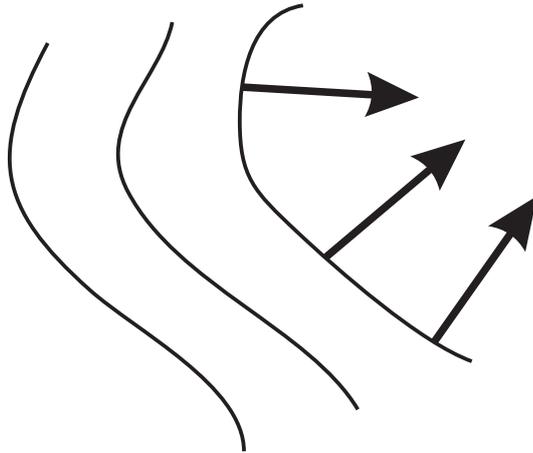
Figure 5.2: The optico-mechanical analogy: motion in configuration space is like motion orthogonal to wavefronts of constant phase.

Recall that Hamilton's principle stated that the classical equations of motion arise from trajectories of motion that extremise the action:

$$S[\mathbf{q}_{true} + \delta\mathbf{q}] = S[\mathbf{q}_{true}] + \mathcal{O}(\delta q^2). \tag{5.33}$$

That is, small variations in trajectory of size δq, lead to variations in the action of the trajectory of size δq^2. We can ask however: what about trajectories away from the true path? Do these have any physical significance? In classical mechanics, the answer is no: the system evolves along the true trajectory alone, and it is this least-action trajectory alone that sets the motion. In quantum mechanics however, the answer is yes. In this case it turns out that actually *all* trajectories contribute to the evolution.
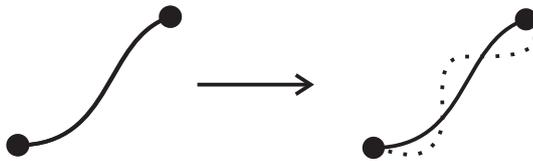
Figure 5.3: The true path, and small deformations from it

Recall that quantum mechanics involves *amplitudes*. We specify the initial state $|I\rangle$, evolve it forward in time, and then ask what is the overlap of the evolved state with a specified final state $\langle F|$. This is what quantum-mechanical evolution is - it is these amplitudes that tell us how states evolve in time, and it is these amplitudes that are the analogues of the classical trajectory. They are

the target of our calculation. That is, we want to compute the amplitudes

$$\mathcal{A} = \langle f | T \exp\left(\frac{-i}{\hbar} \int \hat{H}(t)dt\right) |i\rangle,$$

where T denotes time-ordering. These amplitudes provide all the information about the quantum system, and define the information we can extract.

Also recall that in quantum mechanics position and momentum are conjugate variables: we cannot measure them independently. So when computing amplitudes, we can use either a position basis or a momentum basis to evaluate them: either choice is equally good. If we use the position basis, then solving the quantum evolution equations is equivalent to finding the amplitudes

$$\langle \mathbf{q}_f, t_f | \mathbf{q}_i, t_i \rangle,$$

i.e. the quantum amplitude to go from well-defined position $\mathbf{q}_i$ at time t_i, to well-defined position $\mathbf{q}_f$ at time t_f.

How is this amplitude computed? Recall that in quantum mechanics you cannot ask about what you do not measure, and you do not measure which path the particle took. For all you know, the particle took any one of several paths to reach $\mathbf{q}_f$.

At this point it is useful to recall the double slit experiment - a source of photons incident on a screen and passing through two slits generates interference fringes, even when the source is so weak that only one photon can ever be emitted at a time. This comes from the amplitude to pass through one slit interfering with the amplitude to pass through the other slit. The interference fringes comes from adding these amplitudes: when the amplitudes are in phase, there is constructive interference, and when the amplitudes are out of phase there is destructive interference.
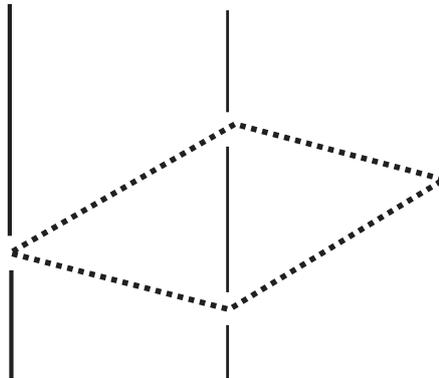

Figure 5.4: Two slits

Given the double slit experiment, we can then imagine extending it to a triple slit experiment. In this case we would have interference from all three slits, and we would have to combine the amplitudes for the particle to pass through each one of the three slits.
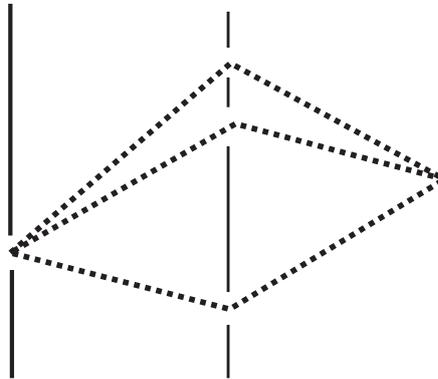

Figure 5.5: Three slits

We can then imagine extending it again, so there are now multiple gratings inbetween the source and the screen. We now have to add up the amplitude for the particle to pass through all possible combinations of slits from each grating.
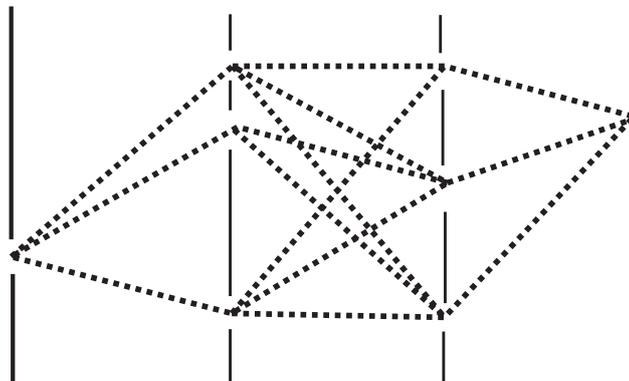
Figure 5.6: Lots of slits and lots of gratings

We carry on extending the number of gratings and the number of slits, until we have an 'infinite' number of gratings and an 'infinite' number of slits. We then again have to combine the amplitudes obtained from passing through all possible combinations of slits from each grating.

However, we should now realise that this final limit of an infinite number of gratings and an infinite number of slits is equivalent to removing all the gratings,

so that we now have photon propagation in free space, and summing over all possible paths from initial position to final position.
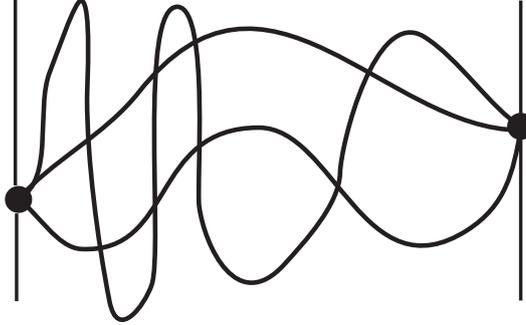

Figure 5.7: An infinite number of slits and an infinite number of gratings

In this limit, the physics of the double slit experiment (different paths interfere) compel us to find the amplitude by summing over the amplitude of all possible paths,

$$\mathcal{A} = \int \mathcal{D}(\text{paths}) \ \text{Amplitude(path)}. \tag{5.34}$$

Feynman realised that the correct prescription here was that the amplitude for any individual path $\mathbf{q}(\mathbf{t})$ is $\exp\left(\frac{iS[\mathbf{q}(t)]}{\hbar}\right)$.

We can then obtain the quantum amplitude as

$$\begin{aligned} \mathcal{A} &= \langle \mathbf{q}_f, t_f | \mathbf{q}_i, t_i \rangle \\ &= \int \mathcal{D}\mathbf{q}(t) \exp\left(\frac{iS[\mathbf{q}(t)]}{\hbar}\right), \end{aligned} \tag{5.35}$$

where we have restricted to paths that start at $\mathbf{q}_i$ and end at $\mathbf{q}_f$. We will state that this works (a derivation is beyond the scope of this course), and that this does indeed fully reproduce 'conventional' Hamiltonian quantum mechanics in its entirety. In fact, for many advanced applications such as quantum field theory the Feynman path integral above is far more useful than the Hamiltonian approach.

The Feynman path integral requires that we sum over all possible paths from initial to final state, weighting each path by $\exp\left(\frac{iS}{\hbar}\right)$. The biggest subtlety in this tends to be the measure factor: when we say summing overall paths, what does the measure factors

$$\mathcal{D}\mathbf{q}(t)$$

actually *mean*? How do we actually compute with this? What does it mean to sum over an infinite number of possible paths? Although this is a bit confusing at first, this issue can be understood, eventually making the Feynman path integral one of the most powerful tools in theoretical physics.

However, we can understand the classical limit of the path integral even without worrying about the complicated and subtle issues of the measure. The classical limit is always $\hbar \to 0$, and it is instructive to think about what happens to the path integral as we take this limit $\hbar \to 0$. The path integral gives us a sum of phases, and we know that to get large contributions in such sums, we require the phases to add constructively.

Now, note that small variations in $S[\mathbf{q}(t)]$ of order $2\pi\hbar$ lead to a complete 2π variation in phase of $\exp\left(\frac{i}{\hbar}S\right)$. So as we take the limit $\hbar \to 0$, it follows that even the smallest changes in $S[\mathbf{q}(t)]$ correspond to large changes in phase (and therefore destructive interference). If we schematically expand

$$S[q(t)] = S[q_0(t)] + \delta q \frac{\partial S}{\partial q}, \tag{5.36}$$

and by expanding about q_0 we make the path integral

$$\int \mathcal{D}q \to \int \mathcal{D}(\delta q), \tag{5.37}$$

we can see that for general paths then small variations lead to destructive interference of amplitudes.

There is one exception to this. If we are at a point of stationary phase, then small variations in the path lead to paths which contribute with the *same* phase as the original path. In this case, we obtain constructive interference of amplitudes, as $S[q + \delta q]$ has the same phase as $S[q]$. The constructive interference of these terms imply that they pile up to give a large contribution to the amplitude.

In the strict limit $\hbar \to 0$, the *only* contribution to the amplitude comes from these paths of stationary phase. All other paths have neighbours that destructively interfere, and the phase cycles infinitely rapidly from one path to the next. However, we now realise that we have in fact recovered Hamilton's action principle: paths of stationary phase are the paths that extremise the action, and so in the classical limit $\hbar \to 0$, the only contribution comes from paths that extremise the action: which is precisely the action principle.

We therefore see that the formulation of quantum mechanics in terms of the Feynman path integral provides a way to understand the action principle as the classical limit of the path integral.

5.3   Integrable systems and Action/Angle Variables

We now return to Hamiltonian systems. We recall the general structure of Hamiltonian mechanics,

$$\dot{q}_i \;\; = \;\; \frac{\partial H}{\partial p_i},$$

$$\dot{p}_i \quad = \quad -\frac{\partial H}{\partial q_i}. \tag{5.38}$$

In general this gives rise to some arbitrary flow through $(\mathbf{q}, \mathbf{p})$ phase space. We
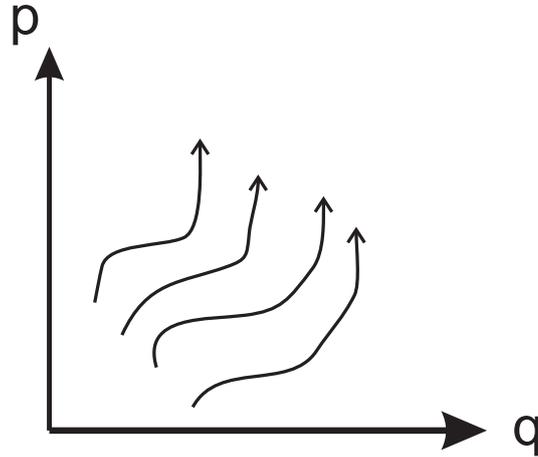

Figure 5.8: Flows in phase space

know however that there are often conserved quantities of the motion, and so the flow is on a restricted surface within phase space. We also know that we can use canonical transformations to change coordinates on phase space. We can therefore ask: can we find coordinates such that we can make the flow extremely simple. In particular, can we find a choice of coordinates $(\mathbf{Q}, \mathbf{P})$ such that the flow equations become

$$
\begin{aligned}
\dot{Q}_i &= \text{constant}, \\
\dot{P}_i &= 0.
\end{aligned}
\tag{5.39}
$$

In this case the coordinates P_i are just constants of the motion, while the coordinates $\mathbf{Q}_i$ just evolve linearly. If they exist, this coordinate system is clearly the right coordinate system to solve the system in.

An *integrable* system is a system for which this redefinition exists, and we are able to find a choice of coordinates such that at all points in phase space the equations of motion take the above very simple form.

More specifically, for a $2n$-dimensional system, $(q_1, \ldots q_n, p_1, \ldots p_n)$, we say that the system is integrable if we can perform a canonical transformation $(\mathbf{q}, \mathbf{p}) \to (\phi, \mathbf{I})$, such that

$$H(\phi, \mathbf{I}) = H(\mathbf{I}), \tag{5.40}$$

with no ϕ dependence. That is, the Hamiltonian depends only on the I_i coordinates, and all ϕ-coordinates are *ignorable* - they do not appear in the Hamiltonian.
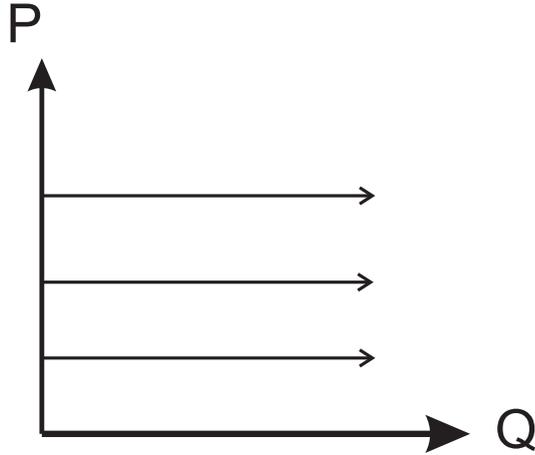
Figure 5.9: What we would like the flow in phase space to look like

In this case we call the ϕ coordinates *angle* variables and the I coordinates *action* variables. When can we do this? We state the following theorem due to Liouville (we do not call this Liouville's theorem to avoid confusion with the previous Liouville's theorem on conservation of volume in phase space).

Suppose we have a Hamiltonian system with n constants of motion I_i, $i = 1, \ldots, n$, such that their Poisson brackets all mutually vanish:

$$[I_i, I_j] = 0.$$

Then the system is *integrable* and a set of action/angle variables exist.

The proof of this is beyond the scope of the course. It can be found for example in the book by Arnold.

If we can succeed in finding action/angle variables, then the evolution equations are

$$\dot{\phi}_i \;\; = \;\; \frac{\partial H}{\partial I_i} = \omega_i, \tag{5.41}$$

$$\dot{I}_i \;\; = \;\; -\frac{\partial H}{\partial \phi_i} = 0. \tag{5.42}$$

and so we have

$$\phi_i \;\; = \;\; \omega_i t + A_i, \tag{5.43}$$
$$I_i \;\; = \;\; B_i, \tag{5.44}$$

where ω_i, A_i and B_i are constants.

For bound systems with a finite range (generally due to energy conservation), the ϕ_i are periodic with a conventional range of 2π (this explains the origin of the name 'angle' variable).
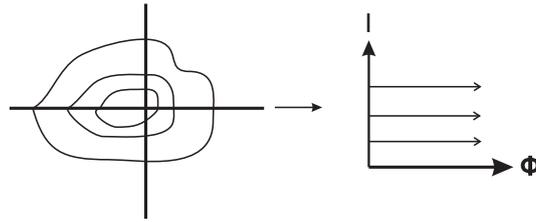
Figure 5.10: The global transformation to action/angle variables

Note that integrability, or the existence of action/angle variables, is a *global* property of the system. *Locally* we can *always* do a transformation of variables to make the system look integrable. The difficulty is in extending this local transformation to a global transformation. Roughly, you can always straighten out a set of flow lines locally - the problem is always to extend this to do this globally.
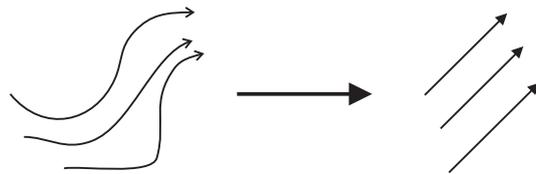

Figure 5.11: The local transformation to action/angle variables: always possible!

You might want to think here about similarities to the problem of combing all the hair on a surface flat. You can do this on a cylinder, but not on a sphere. Locally of course the two surfaces are the same - the difference is global. In a similar way integrability is a global property of a system, not a local one.

We now focus on 1-dimensional systems with a time-independent Hamiltonian. In this case H is a conserved quantity, and so by the above theorem action/angle variables always exist. How do we find them?
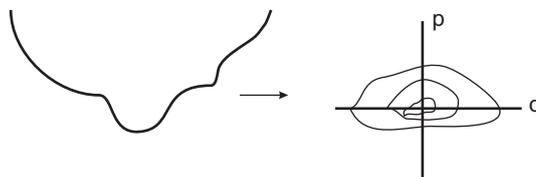

Figure 5.12: A general potential giving rise to periodic motion

Suppose we have periodic motion in a general potential. What are the action/angle variables? We know that I, as a conserved quantity, must be some function of energy, $I = I(H)$.

We will state the correct answer, and then show why it is correct.

$$I = \frac{1}{2\pi} \oint p \, dq, \qquad \phi = \frac{\partial}{\partial I} \int_0^q p \, dq. \qquad (5.45)$$

That is, $I(E)$ is the phase space area enclosed by an orbit of energy E. To verify that this is the correct expression for I, we want to show that $\frac{\partial H}{\partial I} = \omega$, where ω is the frequency of the orbit.

To prove this, we note that for a classical mechanical system

$$p = \sqrt{2m}\sqrt{E - V(q)}, \qquad (5.46)$$

and as we also have $p = m\dot{q}$, we can write

$$dt = \sqrt{\frac{m}{2}} \frac{dq}{\sqrt{E - V(q)}}. \qquad (5.47)$$

If we integrate over one period of the motion, we have

$$
\begin{aligned}
\frac{2\pi}{\omega} = T &= \sqrt{\frac{m}{2}} \oint \frac{dq}{\sqrt{E - V(q)}} \\
&= \oint \sqrt{2m} \left(\frac{d}{dE} \sqrt{E - V(q)} \right) dq \\
&= \frac{d}{dE} \oint \sqrt{2m} \int \sqrt{E - V(q)} \, dq \\
&= \frac{d}{dE} \oint p \, dq \\
&= 2\pi \frac{dI}{dE}.
\end{aligned}
\qquad (5.48)
$$

We therefore have $\omega = \frac{dE}{dI}$, as required for action/angle variables.

Let us illustrate the calculation of action/angle variables by calculating action/angle variables for the 1-dimensional simple harmonic oscillator. The orbits of the harmonic oscillator are well known: they correspond to ellipses in phase space, with a maximal range of q of $\sqrt{\frac{2E}{k}}$ and a maximal range for p of $\sqrt{2mE}$.

We first want to determine the action variable. This is given by the area enclosed under this ellipse, and so

$$2\pi I = \pi \sqrt{\frac{2E}{k}} \sqrt{2mE} = 2\pi E \sqrt{\frac{m}{k}}, \qquad (5.49)$$

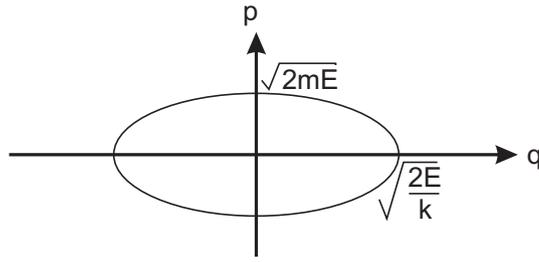which we can write as

$$I = \frac{E}{\omega_0}. \qquad (5.50)$$

Figure 5.13: Orbits for the simple harmonic oscillator

To determine ϕ, we have

$$
\begin{aligned}
\phi = \frac{\partial}{\partial I} \int_0^q p \, dq &= \frac{\partial}{\partial I} \int_0^q \sqrt{2m\omega_0 I - kmq^2} \, dq \\
&= m\omega_0 \int_0^q \frac{dq}{\sqrt{2m\omega_0 I - kmq^2}} \\
&= \arcsin\left(q \sqrt{\frac{m\omega_0}{2I}} \right).
\end{aligned}
\tag{5.51}
$$

This angle variable ϕ then evolves as

$$
\phi = \omega_0 t + \beta,
\tag{5.52}
$$

as an angle variable should.