Path Integrals in Quantum Mechanics and Quantum Field Theory C6, MT 2017

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The parts of the script that are colored blue are not covered in the lecture. They may however be discussed (at least in parts) in the exercises.

The parts of the script that are colored red are strictly non-examinable and will not be covered in the exercises. They represent, for those who are interested, a more advanced gloss on the material.

These notes are lightly modified versions of the lectures notes of Fabian Essler and Uli Haisch.

1 Some Mathematical Background

(Note: this section should be largely revision. It serves to recall the key ideas that we will need to develop the path integral formulation of quantum mechanics and quantum field theory.)

Functional Methods form a central part of modern theoretical physics. In the following we recall the notion of functionals and how to manipulate them.

1.1 Functionals

What is a *functional*? You all know that a real function can be viewed as a *map* from e.g. an interval [a, b] to the real numbers

$$f:[a,b] \to \mathbb{R} , \quad x \to f(x).$$
 (1.1)

A functional is similar to a function in that it maps all elements in a certain domain to real numbers, however, the nature of its domain is very different. Instead of acting on all points of an interval or some other subset of the real numbers, the domain of functionals consists of (suitably chosen) classes of functions. In other words, given some class $\{f\}$ of functions, a functional F is a map

$$F: \{f\} \to \mathbb{R} , \quad f \to F[f].$$
 (1.2)

Examples of functionals are the distance between two points or the action functional,

1. The distance between two points. A very simple functional F consists of the map which assigns to all paths between two fixed points the length of the path. To write this functional explicitly, let us consider a simple two-dimensional situation in the (x, y)plane and choose two points (x_1, y_1) and (x_2, y_2) . We consider the set of paths that do not turn back, i.e. paths along which x increases monotonically as we go from (x_1, y_1) to (x_2, y_2) . These can be described by the set of functions $\{f\}$ on the interval $[x_1, x_2]$ satisfying $f(x_1) = y_1$ and $f(x_2) = y_2$. The length of a path is then given by the wellknown expression

$$F[f(x)] = \int_{x_1}^{x_2} dx' \sqrt{1 + f'(x')^2} .$$
 (1.3)

2. Action Functionals. These are very important in Physics. Let us recall their definition in the context of classical mechanics. Start with n generalised coordinates $\mathbf{q}(t) = (q_1(t), \ldots, q_n(t))$ and a Lagrangian $L = L(\mathbf{q}, \dot{\mathbf{q}})$. Then, the action functional $S[\mathbf{q}]$ is defined by

$$S[\mathbf{q}] = \int_{t_1}^{t_2} dt \, L(\mathbf{q}(t), \dot{\mathbf{q}}(t)) \;. \tag{1.4}$$

It depends on classical paths $\mathbf{q}(t)$ between times t_1 and t_2 satisfying the boundary conditions $\mathbf{q}(t_1) = \mathbf{q}_1$ and $\mathbf{q}(t_2) = \mathbf{q}_2$.

1.2 Functional differentiation

In both the examples given above a very natural question to ask is what function *extremizes* the functional. In the first example this corresponds to wanting to know the path that minimizes the distance between two points. In the second example the extremum of the action functional gives the solutions to the classical equations of motion. This is known as *Hamilton's principle*. In order to figure out what function extremizes the functional it is very useful to generalize the notion of a derivative. For our purposes we define the *functional derivative* by

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon \delta(x - y)] - F[f(x)]}{\epsilon} .$$
(1.5)

Here, as usual, we should think of the δ -function as being defined as the limit of a test function, e.g.

$$\delta(x) = \lim_{a \to 0} \frac{1}{\sqrt{\pi a}} e^{-x^2/a^2} , \qquad (1.6)$$

and take the limit $a \to 0$ only in the end (after commuting the limit with all other operations such as the $\lim_{\epsilon\to 0}$ in 1.5). Importantly, the derivative defined in this way is a linear operation which satisfies the product and chain rules of ordinary differentiation and commutes with ordinary integrals and derivatives. Let us see how functional differentiation works for our two examples. 1. The distance between two points. In analogy with finding stationary points of functions we want to extremize 1.3 by setting its functional derivative equal to zero

$$0 = \frac{\delta F[f(x)]}{\delta f(y)} . \tag{1.7}$$

We first do the calculation by using the definition 1.5.

$$\frac{\delta F[f(x)]}{\delta f(y)} = \lim_{\epsilon \to 0} \int_{x_1}^{x_2} dx' \ \frac{\sqrt{1 + \left[f'(x') + \epsilon \delta'(x' - y)\right]^2} - \sqrt{1 + \left[f'(x')\right]^2}}{\epsilon}.$$
 (1.8)

The Taylor expansion of the square root is $\sqrt{1+2\epsilon} = 1 + \epsilon + \dots$, which gives

$$\sqrt{1 + [f'(x') + \epsilon \delta'(x' - y)]^2} = \sqrt{1 + [f'(x')]^2} + \frac{\epsilon f'(x')\delta'(x' - y)}{\sqrt{1 + [f'(x')]^2}} + \mathcal{O}(\epsilon^2) , \quad (1.9)$$

where $\delta'(x)$ is the derivative of the delta-function and $\mathcal{O}(\epsilon^2)$ denote terms proportional to ϵ^2 . Substituting this back into 1.8 we have ¹

$$\frac{\delta F[f(x)]}{\delta f(y)} = \int_{x_1}^{x_2} dx' \ \frac{\delta'(x'-y)f'(x')}{\sqrt{1+[f'(x')]^2}} = -\frac{d}{dy}\frac{f'(y)}{\sqrt{1+[f'(y)]^2}}.$$
(1.11)

The solution to 1.7 is thus

$$f'(y) = \text{const},\tag{1.12}$$

which describes a straight line. In practice we don't really go back to the definition of the functional derivative any more than we use the definition of an ordinary derivative to work it out, but proceed as follows.

• We first interchange the functional derivative and the integration

$$\frac{\delta F[f(x)]}{\delta f(y)} = \int_{x_1}^{x_2} dx' \ \frac{\delta}{\delta f(y)} \sqrt{1 + [f'(x')]^2}.$$
(1.13)

• Next we use the chain rule

$$\frac{\delta\sqrt{1+f'(x')^2}}{\delta f(y)} = \frac{1}{2\sqrt{1+f'(x')^2}} \frac{\delta(1+f'(x')^2)}{\delta f(y)} = \frac{f'(x')}{\sqrt{1+f'(x')^2}} \frac{\delta f'(x')}{\delta f(y)}.$$
 (1.14)

• Finally we interchange the functional and the ordinary derivative

$$\frac{\delta f'(x')}{\delta f(y)} = \frac{d}{dx'} \frac{\delta f(x')}{\delta f(y)} = \frac{d}{dx'} \delta(x'-y) . \qquad (1.15)$$

The last identity follows from our definition 1.5.

 $^1\mathrm{In}$ the last step we have used

$$\int_{a}^{b} dx' \delta'(x'-y)g(x') = -g'(y) , \qquad (1.10)$$

which can be proved by "integration by parts".

Now we can put everything together and arrive at the same answer 1.11.

2. Next we want to try out these ideas on our second example and extremize the classical action 1.4 in order to obtain the classical equations of motion. We first interchange functional derivative and integration and then use the chain rule to obtain

$$\frac{\delta S[\mathbf{q}]}{\delta q_i(t)} = \frac{\delta}{\delta q_i(t)} \int_{t_1}^{t_2} d\tilde{t} \, L(\mathbf{q}(\tilde{t}), \dot{\mathbf{q}}(\tilde{t})) \tag{1.16}$$

$$= \int_{t_1}^{t_2} d\tilde{t} \left[\frac{\partial L}{\partial q_j} (\mathbf{q}, \dot{\mathbf{q}}) \frac{\delta q_j(\tilde{t})}{\delta q_i(t)} + \frac{\partial L}{\partial \dot{q}_j} (\mathbf{q}, \dot{\mathbf{q}}) \frac{\delta \dot{q}_j(\tilde{t})}{\delta q_i(t)} \right]$$
(1.17)

(1.18)

We now use that $\frac{\delta \dot{q}_j(\tilde{t})}{\delta q_i(t)} = \frac{d}{d\tilde{t}} \frac{\delta q_j(\tilde{t})}{\delta q_i(t)}$ and integrate by parts with respect to \tilde{t}

$$\frac{\delta S[\mathbf{q}]}{\delta q_i(t)} = \int_{t_1}^{t_2} d\tilde{t} \left[\frac{\partial L}{\partial q_j}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{d\tilde{t}} \frac{\partial L}{\partial \dot{q}_j}(\mathbf{q}, \dot{\mathbf{q}}) \right] \frac{\delta q_j(\tilde{t})}{\delta q_i(t)}$$
(1.19)

$$= \int_{t_1}^{t_2} d\tilde{t} \left[\frac{\partial L}{\partial q_j}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{d\tilde{t}} \frac{\partial L}{\partial \dot{q}_j}(\mathbf{q}, \dot{\mathbf{q}}) \right] \delta_{ij} \delta(\tilde{t} - t) = \frac{\partial L}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) \delta_{ij} \delta(\tilde{t} - t) = \frac{\partial L}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) \delta_{ij} \delta(\tilde{t} - t) = \frac{\partial L}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) \delta_{ij} \delta(\tilde{t} - t) = \frac{\partial L}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) \delta_{ij} \delta(\tilde{t} - t) \delta_{ij} \delta(\tilde{t} - t) \delta_{ij} \delta(\tilde{t} - t)$$

In the second last step we have used

$$\frac{\delta q_j(\tilde{t})}{\delta q_i(t)} = \delta_{ij}\delta(\tilde{t} - t) , \qquad (1.21)$$

which follows straightforwardly from our general definition 1.5. Thus we conclude that the extrema of the classical action are given by paths that fulfil the equations of motion

$$\frac{\partial L}{\partial q_i}(\mathbf{q}, \dot{\mathbf{q}}) - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i}(\mathbf{q}, \dot{\mathbf{q}}) = 0.$$
(1.22)

1.3 Multidimensional Gaussian Integrals

As a reminder, we start with a simple one-dimensional Gaussian integral over a single variable y. It is given by

$$I(z) \equiv \int_{-\infty}^{\infty} dy \, \exp(-\frac{1}{2}zy^2) = \sqrt{\frac{2\pi}{z}} \,, \tag{1.23}$$

where z is a complex number with $\operatorname{Re}(z) > 0$. The standard proof of this relation involves writing $I(z)^2$ as a two-dimensional integral over y_1 and y_2 and then introducing two-dimensional polar coordinates $r = \sqrt{y_1^2 + y_2^2}$ and φ . Explicitly,

$$I(z)^{2} = \int_{-\infty}^{\infty} dy_{1} \exp(-\frac{1}{2}zy_{1}^{2}) \int_{-\infty}^{\infty} dy_{2} \exp(-\frac{1}{2}zy_{2}^{2}) = \int_{-\infty}^{\infty} dy_{1} \int_{-\infty}^{\infty} dy_{2} \exp(-\frac{1}{2}z(y_{1}^{2} + (y_{2}^{2}))) dy_{1})$$

$$= \int_0^{2\pi} d\varphi \, \int_0^\infty dr \, r \exp(-\frac{1}{2}zr^2) = \frac{2\pi}{z} \,. \tag{1.25}$$

Next we consider n-dimensional Gaussian integrals

$$W_0(\mathbf{A}) \equiv \int d^n \mathbf{y} \, \exp\left(-\frac{1}{2}\mathbf{y}^T \mathbf{A}\mathbf{y}\right) \,, \qquad (1.26)$$

over variables $\mathbf{y} = (y_1, \ldots, y_n)$, where \mathbf{A} is a symmetric, positive definite matrix (all its eigenvalues are positive). This integral can be reduced to a product of one-dimensional Gaussian integrals by diagonalising the matrix \mathbf{A} . Consider an orthogonal rotation \mathbf{O} such that $\mathbf{A} = \mathbf{O}\mathbf{D}\mathbf{O}^T$ with a diagonal matrix $\mathbf{D} = \text{diag}(a_1, \ldots, a_n)$. The eigenvalues a_i are strictly positive since we have assumed that \mathbf{A} is positive definite. Introducing new coordinates $\tilde{\mathbf{y}} = \mathbf{O}^T \mathbf{y}$ we can write

$$\mathbf{y}^T \mathbf{A} \mathbf{y} = \tilde{\mathbf{y}}^T \mathbf{D} \tilde{\mathbf{y}} = \sum_{i=1}^n a_i \tilde{y}_i^2 , \qquad (1.27)$$

where the property $\mathbf{O}^T \mathbf{O} = \mathbf{1}$ of orthogonal matrices has been used. Note further that the Jacobian of the coordinate change $\mathbf{y} \to \tilde{\mathbf{y}}$ is one, since $|\det(\mathbf{O})| = 1$. Hence, using Eqs. (1.23) and (1.27) we find for the integral (1.26)

$$W_0(\mathbf{A}) = \prod_{i=1}^n \int d\tilde{y}_i \exp(-\frac{1}{2}a_i\tilde{y}_i^2) = (2\pi)^{n/2}(a_1a_2\dots a_n)^{-1/2} = (2\pi)^{n/2}(\det\mathbf{A})^{-1/2} .$$
(1.28)

To summarise, we have found for the multidimensional Gaussian integral (1.26) that

$$W_0(\mathbf{A}) = (2\pi)^{n/2} (\det \mathbf{A})^{-1/2} ,$$
 (1.29)

a result which will be of some importance in the following. We note that if we multiply the matrix **A** by a complex number z with $\operatorname{Re}(z) > 0$ and then follow through exactly the same steps, we find

$$W_0(z\mathbf{A}) = \left(\frac{2\pi}{z}\right)^{n/2} (\det \mathbf{A})^{-1/2} .$$
 (1.30)

One obvious generalisation of the integral (1.26) involves adding a term linear in **y** in the exponent, that is

$$W_0(\mathbf{A}, \mathbf{J}) \equiv \int d^n \mathbf{y} \, \exp\left(-\frac{1}{2}\mathbf{y}^T \mathbf{A}\mathbf{y} + \mathbf{J}^T \mathbf{y}\right) \,. \tag{1.31}$$

Here $\mathbf{J} = (J_1, \ldots, J_n)$ is an *n*-dimensional vector. Changing variables $\mathbf{y} \to \tilde{\mathbf{y}}$, where

$$\mathbf{y} = \mathbf{A}^{-1}\mathbf{J} + \tilde{\mathbf{y}} \tag{1.32}$$

this integral can be written as

$$W_0(\mathbf{A}, \mathbf{J}) = \exp\left(\frac{1}{2}\mathbf{J}^T \mathbf{A}^{-1}\mathbf{J}\right) \int d^n \tilde{\mathbf{y}} \exp\left(-\frac{1}{2}\tilde{\mathbf{y}}^T \mathbf{A}\tilde{\mathbf{y}}\right) .$$
(1.33)

The remaining integral is Gaussian without a linear term, so can be easily carried out using the above results. Hence, one finds

$$W_0(\mathbf{A}, \mathbf{J}) = (2\pi)^{n/2} (\det \mathbf{A})^{-1/2} \exp\left(\frac{1}{2} \mathbf{J}^T \mathbf{A}^{-1} \mathbf{J}\right) .$$
(1.34)

2 Path Integrals in Quantum Mechanics

So far you have encountered two ways of doing QM:

- 1. Following Schrödinger, we can solve the Schrödinger equation for the wave function \rightarrow Fun with PDEs...
- 2. Following Heisenberg, we can work with operators, commutation relations, eigenstates \rightarrow Fun with Linear Algebra...

Historically it took some time to realize that these were in fact equivalent (this was one of Dirac's contributions). In their own words: I knew of Heisenberg's theory, of course, but I felt discouraged, not to say repelled, by the methods of transcendental algebra, which appeared difficult to me, and by the lack of visualizability. (Schrödinger in 1926)

The more I think about the physical portion of Schrödinger's theory, the more repulsive I find it. What Schrödinger writes about the visualizability of his theory is probably not quite right, in other words it's crap. (Heisenberg, writing to Pauli in 1926)

There is a third approach to QM - also equivalent - due to *Feynman*. He developed it first as a graduate student, inspired by a mysterious remark in a paper by Dirac. Those were the days! Feynman's approach is particularly useful for QFTs and many-particle QM problems, as it makes certain calculations much easier.

In particular, we have seen in the previous field theory parts of the lectures the role of the propagator - the amplitude to propagate from a certain initial configuration to a certain final configuration. This propagator is crucial for scattering questions. We first work out what this is using the Heisenberg/Schrödinger formulation and then formulate QM à la Feynman.

2.1 The Propagator

Our starting point is the time-dependent Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle.$$
 (2.35)

We recall that the wave function is given by

$$\psi(\vec{x},t) = \langle \vec{x} | \psi(t) \rangle. \tag{2.36}$$

Eqn 2.35 can be integrated to give

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar}Ht}|\psi(0)\rangle \tag{2.37}$$

The time-evolution operator in QM is thus (assuming that H is time-independent)

$$U(t;t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}.$$
(2.38)

A central object in Feynman's approach is the *propagator*

$$\langle \vec{x}' | U(t; t_0) | \vec{x} \rangle, \tag{2.39}$$

where $|\vec{x}\rangle$ are the simultaneous eigenstates of the position operators \hat{x} , \hat{y} and \hat{z} . The propagator is the probability amplitude for finding our QM particle at position \vec{x}' at time t, if it started at position \vec{x} at time t_0 . To keep notations simple, we now consider a particle moving in one dimension with time-independent Hamiltonian

$$H = \hat{T} + \hat{V} = \frac{\hat{p}^2}{2m} + V(\hat{x}).$$
(2.40)

We want to calculate the propagator

$$\langle x_N | U(t;0) | x_0 \rangle. \tag{2.41}$$

It is useful to introduce small time steps

$$t_n = n\epsilon , \quad n = 0, \dots, N, \tag{2.42}$$

where $\epsilon = t/N$. Then we have by construction

$$U(t;0) = \left(e^{-\frac{i}{\hbar}H\epsilon}\right)^{N}.$$
(2.43)

The propagator is

$$\langle x_N | U(t;0) | x_0 \rangle = \langle x_N | e^{-\frac{i}{\hbar}H\epsilon} \cdots e^{-\frac{i}{\hbar}H\epsilon} | x_0 \rangle$$

$$= \int dx_{N-1} \cdots \int dx_1 \ \langle x_N | e^{-\frac{i}{\hbar}H\epsilon} | x_{N-1} \rangle \langle x_{N-1} | e^{-\frac{i}{\hbar}H\epsilon} | x_{N-2} \rangle \cdots \langle x_1 | e^{-\frac{i}{\hbar}H\epsilon} (\mathbf{2}_0)$$

where we have inserted N-1 resolutions of the identity in terms of position eigenstates

$$\mathbf{1} = \int dx \, |x\rangle \langle x| \,. \tag{2.45}$$

This expression now has a very nice and intuitive interpretation, see Fig. 2.1: The propagator, i.e. the probability amplitude for finding the particle at position x_N and time t given that it was at position x_0 at time 0 is given by the sum over all "paths" going from x_0 to x_N (as x_1, \ldots, x_{N-1} are integrated over).

In the next step we determine the "infinitesimal propagator"

$$\langle x_{n+1}|e^{-\frac{i}{\hbar}H\epsilon}|x_n\rangle. \tag{2.46}$$

Importantly we have $[\hat{T}, \hat{V}] \neq 0$ and concomitantly

$$e^{\alpha(\hat{T}+\hat{V})} \neq e^{\alpha\hat{T}}e^{\alpha\hat{V}}.$$
(2.47)

However, using that ϵ is infinitesimal, we have

$$e^{-\frac{i}{\hbar}\epsilon(\hat{T}+\hat{V})} = 1 - \frac{i}{\hbar}\epsilon(\hat{T}+\hat{V}) + \mathcal{O}(\epsilon^2) ,$$



Figure 2.1: Propagator as sum over paths.

$$e^{-\frac{i}{\hbar}\epsilon\hat{T}}e^{-\frac{i}{\hbar}\epsilon\hat{V}} = 1 - \frac{i}{\hbar}\epsilon(\hat{T} + \hat{V}) + \mathcal{O}(\epsilon^2).$$
(2.48)

So up to terms of order ϵ^2 we have

$$\langle x_{n+1}|e^{-\frac{i}{\hbar}H\epsilon}|x_n\rangle \simeq \langle x_{n+1}|e^{-\frac{i}{\hbar}\hat{T}\epsilon}e^{-\frac{i}{\hbar}\hat{V}\epsilon}|x_n\rangle = \langle x_{n+1}|e^{-\frac{i}{\hbar}\hat{T}\epsilon}|x_n\rangle e^{-\frac{i}{\hbar}V(x_n)\epsilon}, \qquad (2.49)$$

where we have used that $\hat{V}|x\rangle = V(x)|x\rangle$. As $\hat{T} = \hat{p}^2/2m$ it is useful to insert a complete set of momentum eigenstates ² to calculate

$$\langle x_{n+1}|e^{-\frac{i}{\hbar}\hat{T}\epsilon}|x_n\rangle = \int \frac{dp}{2\pi\hbar} \langle x_{n+1}|e^{-\frac{ip^2\epsilon}{2m\hbar}}|p\rangle \langle p|x_n\rangle = \int \frac{dp}{2\pi\hbar} e^{-\frac{ip^2\epsilon}{2m\hbar} - i\frac{p}{\hbar}(x_n - x_{n+1})}$$
$$= \sqrt{\frac{m}{2\pi\hbar\epsilon}} e^{\frac{im}{2\hbar\epsilon}(x_n - x_{n+1})^2}.$$
(2.50)

In the second step we have used that $\hat{p}|p\rangle=p|p\rangle$ and that

$$\langle x|p\rangle = e^{\frac{ipx}{\hbar}}.\tag{2.51}$$

The integral over p is performed by changing variables to $p' = p + \frac{m}{\epsilon}(x_n - x_{n+1})$ (and giving ϵ a very small imaginary part in order to make the integral convergent). Substituting 2.50 and 2.49 back into our expression 2.44 for the propagator gives

$$\langle x_N | U(t;0) | x_0 \rangle = \lim_{N \to \infty} \left[\frac{m}{2\pi i \hbar \epsilon} \right]^{\frac{N}{2}} \int dx_1 \dots dx_{N-1} \exp\left(\frac{i\epsilon}{\hbar} \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \right) dx_{N-1} \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)$$

Note that in this expression there are no operators left.

²We use a normalization $\langle p|k\rangle = 2\pi\hbar\delta(p-k)$, so that $\mathbf{1} = \int \frac{dp}{2\pi\hbar} |p\rangle\langle p|$.

2.1.1 Propagator as a "Functional Integral"

The way to think about 2.52 is as a sum over trajectories:

- x_0, \ldots, x_N constitute a discretization of a path x(t'), where we set $x_n \equiv x(t_n)$.
- We then have

$$\frac{x_{n+1} - x_n}{\epsilon} = \frac{x(t_{n+1}) - x(t_n)}{t_{n+1} - t_n} \simeq \dot{x}(t_n), \qquad (2.53)$$

and

$$\epsilon \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 - V(x_n) \simeq \int_0^t dt' \left[\frac{m}{2} \dot{x}^2(t') - V(x) \right] \equiv \int_0^t dt' \mathcal{L}[\dot{x}, x], \quad (2.54)$$

where \mathcal{L} is the Lagrangian of the system. In classical mechanics the time-integral of the Lagrangian is known as the *action*

$$S = \int_0^t dt' \mathcal{L}.$$
 (2.55)

- The integral over $x_1, \ldots x_{N-1}$ becomes a functional integral, also known as a path integral, over all paths x(t') that start at x_0 at time t' = 0 and end at x_N at time t' = t.
- The prefactor in 2.52 gives rise to an overall (infinite) normalization and we will denote it by \mathcal{N} .

These considerations lead us to express the propagator as the following *formal expression*

$$\langle x_N | U(t;0) | x_0 \rangle = \mathcal{N} \int \mathcal{D}x(t') e^{\frac{i}{\hbar}S[x(t')]}.$$
(2.56)

What is in fact meant by 2.56 is the limit of the discretized expression 2.52. The ultimate utility of 2.56 is that it provides a compact notation, that on the one hand will allow us to manipulate functional integrals, and on the other hand provides a nice, intuitive interpretation.

2.2 Quantum Mechanics à la Feynman

Feynman's formulation of Quantum Mechanics is based on the single postulate that the probability amplitude for propagation from a position x_0 to a position x_N is obtained by summing over all possible paths connecting x_0 and x_N , where each path is *weighted* by a phase factor $\exp\left(\frac{i}{\hbar}S\right)$, where S is the *classical action* of the path. This provides a new way of thinking about QM!

2.3 Classical Limit and Stationary Phase Approximation

An important feature of 2.56 is that it gives us a nice way of thinking about the classical limit " $\hbar \rightarrow 0$ " (more precisely in the limit when the dimensions, masses, times etc are so large that the action is huge compared to \hbar). To see what happens in this limit let us first consider the simpler case of an ordinary integral

$$g(a) = \int_{-\infty}^{\infty} dt \ h_1(t) e^{iah_2(t)}, \tag{2.57}$$

when we take the real parameter a to infinity. In this case the integrand will oscillate wildly as a function of t because the phase of exp $(iah_2(t))$ will vary rapidly. The dominant contribution will arise from the points where the phase changes slowly, which are the stationary points

$$h_2'(t) = 0. (2.58)$$

The integral can then be approximated by expanding around the stationary points. Assuming that there is a single stationary point at t_0

$$g(a \gg 1) \approx \int_{-\infty}^{\infty} dt \left[h_1(t_0) + (t - t_0) h_1'(t_0) + \ldots \right] e^{iah_2(t_0) + i\frac{ah_2''(t_0)}{2}(t - t_0)^2},$$
(2.59)

Changing integration variables to $t' = t - t_0$ (and giving *a* a small imaginary part to make the integral converge at infinity) as obtain a Gaussian integral that we can take using 1.23

$$g(a \gg 1) \approx \sqrt{\frac{2\pi i}{ah_2''(t_0)}} h_1(t_0) e^{iah_2(t_0)}$$
 (2.60)

Subleading contributions can be evaluated by taking higher order contributions in the Taylor expansions into account. If we have several stationary points we sum over their contributions. The method we have just discussed is known as *stationary phase approximation*.

The generalization to path integrals is now clear: in the limit $\hbar \to 0$ the path integral is dominated by the vicinity of the *stationary points* of the action S

$$\frac{\delta S}{\delta x(t')} = 0. \tag{2.61}$$

The condition 2.61 precisely defines the classical trajectories x(t')!

2.4 The Propagator for Free Particles

We now wish to calculate the functional integral 2.56 for a free particle, i.e.

$$V(x) = 0. (2.62)$$

Going back to the explicit expression 2.52 we have

$$\langle x_N | U(t;0) | x_0 \rangle = \lim_{N \to \infty} \left[\frac{m}{2\pi i \hbar \epsilon} \right]^{\frac{N}{2}} \int dx_1 \dots dx_{N-1} \exp\left(\frac{i\epsilon}{\hbar} \sum_{n=0}^{N-1} \frac{m}{2} \left(\frac{x_{n+1} - x_n}{\epsilon} \right)^2 \right)$$
(2.63)

It is useful to change integration variables to

$$y_j = x_j - x_N$$
, $j = 1, \dots, N - 1$, (2.64)

which leads to an expression

$$\langle x_N | U(t;0) | x_0 \rangle = \lim_{N \to \infty} \left[\frac{m}{2\pi i \hbar \epsilon} \right]^{\frac{N}{2}} \int d\mathbf{y} \exp\left(-\frac{1}{2} \mathbf{y}^T \mathbf{A} \mathbf{y} + \mathbf{J}^T \cdot \mathbf{y} \right) e^{\frac{im}{2\hbar \epsilon} (x_0 - x_N)^2}.$$
 (2.65)

Here

$$\mathbf{J}^T = \left(\frac{im}{\hbar\epsilon}(x_N - x_0), 0, \dots, 0\right),\tag{2.66}$$

and **A** is a $(N-1) \times (N-1)$ matrix with elements

•

$$A_{jk} = \frac{-im}{\epsilon\hbar} \left[2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1} \right].$$
(2.67)

For a given N 2.65 is a multidimensional Gaussian integral and can be carried out using 1.34

$$\langle x_N | U(t;0) | x_0 \rangle = \lim_{N \to \infty} \left[\frac{m}{2\pi i \hbar \epsilon} \right]^{\frac{N}{2}} (2\pi)^{\frac{N-1}{2}} \left[\det(\mathbf{A}) \right]^{-\frac{1}{2}} \exp\left(\frac{1}{2} \mathbf{J}^T \mathbf{A}^{-1} \mathbf{J} \right) e^{\frac{im}{2\hbar \epsilon} (x_0 - x_N)^2} (2.68)$$

The matrix \mathbf{A} is related to the one dimensional lattice Laplacian, see below. Given the eigenvalues and eigenvectors worked out below we can calculate the determinant and inverse of \mathbf{A} (homework problem). Substituting the results into 2.68 gives

$$\langle x_N | U(t;0) | x_0 \rangle = \sqrt{\frac{m}{2\pi i \hbar t}} e^{\frac{im}{2\hbar t} (x_0 - x_N)^2}.$$
 (2.69)

For a free particle we can evaluate the propagator directly in a much simpler way.

$$\langle x_N | U(t;0) | x_0 \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} \langle x_N | e^{-i\frac{\vec{p}^2 t}{2m\hbar}} | p \rangle \langle p | x_0 \rangle = \int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{-i\frac{p^2 t}{2m\hbar} - i\frac{p(x_0 - x_N)}{\hbar}}$$
$$= \sqrt{\frac{m}{2\pi\hbar\hbar t}} e^{\frac{im}{2\hbar t}(x_0 - x_N)^2}.$$
(2.70)

The matrix A is related to the one dimensional *Lattice Laplacian*. Consider functions of a variable $z_0 \leq z \leq z_N$ with "hard-wall boundary conditions"

$$f(z_0) = f(z_N) = 0. (2.71)$$

The Laplace operator D acts on these functions as

$$Df \equiv \frac{d^2 f(z)}{dz^2}.$$
(2.72)

Discretizing the variable z by introducing N-1 points

$$z_n = z_0 + na_0$$
, $n = 1, \dots, N-1$ (2.73)

where $a_0 = (z_N - z_0)/N$ is a "lattice spacing", maps the function f(z) to a N - 1 dimensional vector

$$f(z) \to \mathbf{f} = (f(z_1), \dots, f(z_{N-1})).$$
 (2.74)

Recalling that

$$\frac{d^2 f}{dz^2}(z) = \lim_{a_0 \to 0} \frac{f(z+a_0) + f(z-a_0) - 2f(z)}{a_0^2} , \qquad (2.75)$$

we conclude that the Lapacian is discretized as follows

$$Df \to a_0^{-2} \Delta \mathbf{f}$$
, (2.76)

where

$$\Delta_{jk} = \delta_{j,k+1} + \delta_{j,k-1} - 2\delta_{j,k}.$$
 (2.77)

Our matrix **A** is equal to $\frac{im}{\epsilon\hbar}\Delta$. The eigenvalue equation

$$\Delta \mathbf{a}_n = \lambda_n \mathbf{a}_n, \quad n = 1, \dots, N - 1 \tag{2.78}$$

gives rise to a recurrence relation for the components $\mathbf{a}_{n,j}$ of \mathbf{a}_n

$$a_{n,j+1} + a_{n,j-1} - (2 + \lambda_n)a_{n,j} = 0.$$
(2.79)

The boundary conditions $a_{n,N} = a_{n,0} = 0$ suggest the ansatz

$$a_{n,j} = C_n \sin\left(\frac{\pi n j}{N}\right). \tag{2.80}$$

Substituting this in to 2.79 gives

$$\lambda_n = 2\cos\left(\frac{\pi n}{N}\right) - 2$$
, $n = 1, \dots, N - 1.$ (2.81)

The normalized eigenvectors of Δ are

$$\mathbf{a}_{n} = \frac{1}{\sqrt{\sum_{j=1}^{N-1} \sin^{2}\left(\frac{\pi n j}{N}\right)}} \begin{pmatrix} \sin\left(\frac{\pi n}{N}\right) \\ \sin\left(\frac{2\pi n}{N}\right) \\ \vdots \\ \sin\left(\frac{\pi (N-1)n}{N}\right) \end{pmatrix}} = \sqrt{\frac{2}{N}} \begin{pmatrix} \sin\left(\frac{\pi n}{N}\right) \\ \sin\left(\frac{2\pi n}{N}\right) \\ \vdots \\ \sin\left(\frac{\pi (N-1)n}{N}\right) \end{pmatrix}$$
(2.82)

3 Path Integrals in Quantum Field Theory

We now want to discuss path integrals within quantum field theory.

3.1 QM Flashback

It has already been explained how QM can be formulated in terms of path integrals. One important finding was that the matrix element between two position eigenstates is given by

$$\langle x, t | x', t' \rangle = \langle x | e^{-iH(t-t')} | x' \rangle \propto \int \mathcal{D}x \, \exp\left(i \int_{t'}^t dt'' \mathcal{L}(x, \dot{x})\right) \,.$$
(3.1)

Notice that here $|x,t\rangle = e^{iHt} |x\rangle_S$ and $|x',t'\rangle = e^{iHt'} |x'\rangle_S$ are Heisenberg picture states.

We also learned in the lectures on interacting quantum fields that the central objects to compute in quantum field theory (QFT) are vacuum expectation values (VEVs) of timeordered field-operator products such as the Feynman propagator

$$D_F(x-y) = \langle 0 | T\phi(x)\phi(y) | 0 \rangle .$$
(3.2)

In QM the analog of (3.2) is simply³

$$\langle x_f, t_f | T \hat{x}(t_1) \hat{x}(t_2) | x_i, t_i \rangle.$$
(3.3)

Focusing on the case $t_1 > t_2$ and inserting complete sets of states we can write the latter expression as

$$\langle x_f, t_f | \hat{x}(t_1) \hat{x}(t_2) | x_i, t_i \rangle = \langle x_f | e^{-iH(t_f - t_1)} \hat{x}_S e^{-iH(t_1 - t_2)} \hat{x}_S e^{-iH(t_2 - t_i)} | x_i \rangle$$

$$= \int dx_1 dx_2 \langle x_f | e^{-iH(t_f - t_1)} | x_1 \rangle \langle x_1 | \hat{x}_S e^{-iH(t_1 - t_2)} | x_2 \rangle \langle x_2 | \hat{x}_S e^{-iH(t_2 - t_i)} | x_i \rangle .$$

$$(3.4)$$

Using now that $\hat{x}|x\rangle = x|x\rangle$, replacing all three expectation values by (3.1) and combining the three path integrals with the integrations over x_1 and x_2 into a single path integral the result (3.4) simplifies further. One finds that

$$\langle x_f, t_f | T \hat{x}(t_1) \hat{x}(t_2) | x_i, t_i \rangle \propto \int \mathcal{D}x \, x(t_1) x(t_2) \, \exp\left(i \int_{t_i}^{t_f} dt \mathcal{L}(x, \dot{x})\right) \,.$$
 (3.5)

For $t_2 > t_1$ the same result holds, because time ordering is automatic in the path-integral formulation. It should also be clear that results similar to (3.5) apply for a product of an arbitrary number of operators \hat{x} . Furthermore, it can be shown that

$$\lim_{t_{i,f}\to\mp\infty} \langle x_f, t_f | T \left(\hat{x}(t_1) \dots \hat{x}(t_n) \right) | x_i, t_i \rangle \propto \langle 0 | T \left(\hat{x}(t_1) \dots \hat{x}(t_n) \right) | 0 \rangle.$$
(3.6)

Therefore one arrives at

$$\langle 0 | T (\hat{x}(t_1) \dots \hat{x}(t_n)) | 0 \rangle \propto \int \mathcal{D}x \, x(t_1) \dots x(t_n) \, e^{iS[x]},$$
(3.7)

with S[x] the action functional.

³In this part of the lecture, we will use hats to distinguish operators from their classical counterparts which appear in the path integral.

3.2 Basics of QFT Path Integrals

In order to keep the following discussion as simple as possible we will focus on the real scalar field ϕ . An extension to more complicated theories would however be straightforward. As we saw the Green's functions of the form

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n) = \left\langle 0 \left| T\left(\hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \right) \right| 0 \right\rangle, \qquad (3.8)$$

play an important role in QFT. In analogy to (3.7) these objects can be written as

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n) = \mathcal{N} \int \mathcal{D}\phi \,\phi(t_1)\ldots\phi(t_n) \,e^{iS[x]} \,, \tag{3.9}$$

where \mathcal{N} is a normalization constant.

Like in QM, we introduce the generating functional

$$W[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{i \int d^4x \left[\mathcal{L}(\phi, \partial_\mu \phi) + J(x)\phi(x)\right]\right\}, \qquad (3.10)$$

for the Green's functions such that

$$\mathcal{G}^{(n)}(x_1, \dots, x_n) = \left. \frac{i^{-n} \,\delta^n W[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0} \,.$$
(3.11)

The value of \mathcal{N} is fixed by requiring that

$$W[J]|_{J=0} = W[0] = \langle 0|0\rangle = 1.$$
 (3.12)

Recalling that in QM a second generating functional called Z[J] has been introduced, we also define

$$Z[J] = -i \ln W[J].$$
 (3.13)

By apply n functional differentiations to Z[J] we get another type of Green's functions⁴

$$G^{(n)}(x_1, \dots, x_n) = \left. \frac{i^{1-n} \,\delta^n Z[J]}{\delta J(x_1) \dots \delta J(x_n)} \right|_{J=0} \,, \tag{3.14}$$

which correspond to *connected* Feynman diagrams, so that it makes sense to call Z[J] the generating functional for connected Green's functions. Notice that all information on the QFT is now encoded in the generating functionals, which are hence the primary objects to calculate. We will do this below for the simplest case of a free real scalar field.

 $^{^4\}mathrm{Realize}$ that these Green's functions are the ones we meet already in (1.65) and (1.66) of the script "Interacting Quantum Fields".

3.3 Generating Functionals For Free Real Scalar

For the purpose of explicit calculations it turns out to be useful to introduce a *Euclidean* or *Wick-rotated* version $W_E[J]$ of the generating functional. To do this we define Euclidean 4-vectors $\bar{x} = (\bar{x}_0, \bar{x}) = (ix_0, x)$, associated derivatives $\bar{\partial}^{\mu} = \partial/\partial \bar{x}_{\mu}$, and an Euclidean version of the Lagrangian, $\mathcal{L}_E = \mathcal{L}_E(\phi, \bar{\partial}_{\mu}\phi)$. To give an example,

$$\mathcal{L} = \frac{1}{2} (\partial_{\mu} \phi)^2 - \frac{1}{2} m^2 \phi^2 \to \mathcal{L}_E = -\frac{1}{2} (\bar{\partial}_{\mu} \phi)^2 - \frac{1}{2} m^2 \phi^2.$$
(3.15)

Starting from (3.10) it is easy to see that the Euclidean version of W[J] is

$$W_E[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{i \int d^4 \bar{x} \left[\mathcal{L}_E(\phi, \bar{\partial}_\mu \phi) + J(\bar{x})\phi(\bar{x})\right]\right\} \,. \tag{3.16}$$

The corresponding Euclidean Green's function are then obtained by

$$\mathcal{G}_E^{(n)}(\bar{x}_1,\ldots,\bar{x}_n) = \left. \frac{i^{-n} \,\delta^n W_E[J]}{\delta J(\bar{x}_1)\ldots\delta J(\bar{x}_n)} \right|_{J=0} \,. \tag{3.17}$$

We now want to derive W[J] for the real Klein-Gordon theory. We start by writing

$$\int d^4 \bar{x} \left(\bar{\partial}_\mu \phi(\bar{x}) \right) \left(\bar{\partial}^\mu \phi(\bar{x}) \right) = \int d^4 \bar{x} \, d^4 \bar{y} \, \phi(\bar{y}) \left(\bar{\partial}^y_\mu \bar{\partial}^\mu_x \delta^{(4)}(\bar{x} - \bar{y}) \phi(\bar{x}) \right) \,. \tag{3.18}$$

It follows that

$$W_E[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\left\{-\frac{1}{2} \int d^4 \bar{x} d^4 \bar{y} \, \phi(\bar{y}) A(\bar{y}, \bar{x}) \phi(\bar{x}) + \int d^4 \bar{x} \, J(\bar{x}) \phi(\bar{x})\right\}, \qquad (3.19)$$

with

$$A(\bar{y},\bar{x}) = \left(\bar{\partial}^y_\mu \bar{\partial}^\mu_x + m^2\right) \delta^{(4)}(\bar{x} - \bar{y}).$$
(3.20)

This is a *Gaussian path integral* with a source J of precisely the type you have discussed in the QM context in the solid-state part of this lecture.

Given this analogy we perform a variable transformation to find an explicit expression for (3.19). Skipping over the details of the actual calculation, one obtains

$$W_E[J] = \bar{\mathcal{N}} \exp\left\{\frac{1}{2} \int d^4 \bar{x} \, d^4 \bar{y} \, J(\bar{y}) D_F^E(\bar{y} - \bar{x}) J(\bar{x})\right\} \,, \tag{3.21}$$

with

$$D_F^E(\bar{y} - \bar{x}) = A^{-1}(\bar{y} - \bar{x}), \qquad (3.22)$$

and $\overline{\mathcal{N}}$ an appropriate normalization.

Fine, but how do we calculate the inverse of the operator A? The idea is to use Fourier transformations and then to go back to Minkowski space. We first recall that

$$\delta^{(4)}(\bar{x} - \bar{y}) = \int \frac{d^4 \bar{p}}{(2\pi)^4} e^{i\bar{p}(\bar{x} - \bar{y})}, \qquad (3.23)$$

which we use to write

$$A(\bar{y},\bar{x}) = \left(\bar{\partial}^y_{\mu}\bar{\partial}^{\mu}_{x} + m^2\right)\delta^{(4)}(\bar{x}-\bar{y}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \left(\bar{p}^2 + m^2\right)e^{i\bar{p}(\bar{x}-\bar{y})}.$$
 (3.24)

Now we invert A by taking the inverse inside the Fourier transformation, i.e.⁵

$$A^{-1}(\bar{y} - \bar{x}) = D_F^E(\bar{y} - \bar{x}) = \int \frac{d^4\bar{p}}{(2\pi)^4} \frac{1}{\bar{p}^2 + m^2} e^{i\bar{p}(\bar{x} - \bar{y})}.$$
(3.26)

To go to Minkowski space we introduce $p = (p_0, \mathbf{p}) = (i\bar{p}_0, \bar{\mathbf{p}})$. Putting things together one finds for the generating functional in Minkowski space

$$W[J] = \exp\left\{-\frac{1}{2}\int d^4x \, d^4y \, J(y)D_F(y-x)J(x)\right\},$$
(3.27)

where

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\epsilon} e^{-ip(x-y)}, \qquad (3.28)$$

is our *Feynman propagator*. Notice that we have chosen $\overline{\mathcal{N}} = 1$ so that (3.12) holds.

From (3.27) we can now derive Green's functions effortless. *E.g.*, for the 2-point function we get from (3.11)

$$\mathcal{G}^{(2)}(x,y) = -\frac{\delta^2 W[J]}{\delta J(x) \delta J(y)} \bigg|_{J=0} = D_F(x-y), \qquad (3.29)$$

which agrees with the result that we got using *canonical quantization*.

From (3.13) and (3.27) we also find

$$Z[J] = \frac{i}{2} \int d^4x \, d^4y \, J(y) D_F(y-x) J(x) \,, \tag{3.30}$$

for the generating functional Z[J] of connected Green's functions. It is important to bear in mind that the results (3.27) and (3.30) hold for the free field theory only.

3.4 Effective Action

Path integrals also provide an intuitive picture for the transition between classical and quantum physics. In order to illustrate this property we define the *classical field* ϕ_c by

$$\phi_c(x) = \frac{\delta Z[J]}{\delta J(x)}.$$
(3.31)

$$\int d^4 y A(x,y) A^{-1}(y,z) = \delta^4(x-z), \qquad (3.25)$$

and that this is precisely how the quoted expression for the inverse behaves.

⁵To see that this works, note that we want

We have

$$\phi_c(x) = \frac{\delta}{\delta J(x)} \left(-i \ln W[J] \right) = -\frac{i}{W[J]} \frac{\delta W[J]}{\delta J(x)} = \frac{\langle 0|\hat{\phi}(x)|0\rangle_J}{\langle 0|0\rangle_J} \,. \tag{3.32}$$

Here we have defined VEVs in the presence of J as follows

$$\langle 0|0\rangle_J = W[J], \qquad \langle 0|\hat{\phi}(x)|0\rangle_J = -i\frac{\delta W[J]}{\delta J(x)}.$$
(3.33)

The final result for ϕ_c shows that the classical field is the suitably normalized VEV of the field operator $\hat{\phi}$, which from a physical standpoint sounds quite reasonable. Recall also that $W[J] = \exp(iZ[J])$ which suggest that the generating functional of connected Green's functions is something like the action in our path integrals $\int \mathcal{D}\phi \dots e^{iS[\phi]}$. This suggest that Z[J] is some sort of effective action.

To remove the effect of the source term that is present in Z[J] we use a Legendre transform. We define the *effective action* as

$$\Gamma[\phi_c] = Z[J] - \int d^4x \, J(x)\phi_c(x) \,. \tag{3.34}$$

In fact, with this definition one has

$$\frac{\delta\Gamma[\phi_c]}{\delta J(y)} = \frac{\delta Z[J]}{\delta J(y)} - \int d^4x \, \frac{\delta J(x)}{\delta J(y)} \, \phi_c(x) = \phi_c(y) - \int d^4x \, \delta^{(4)}(x-y) \, \phi_c(x) = 0 \,, \tag{3.35}$$

so $\Gamma[\phi_c]$ is independent of the source J.

To further see that the definition (3.34) is meaningful, let us discuss the free field case. We begin by deriving an explicit expression for the classic field:

$$\phi_c(x) = \frac{\delta}{\delta J(x)} \frac{i}{2} \int d^4 y d^4 z J(y) D_F(y-z) J(z)$$

= $\frac{i}{2} \left\{ \int d^4 y d^4 z \left[\delta^{(4)}(x-y) D_F(y-z) J(z) + \delta^{(4)}(x-z) D_F(y-z) J(y) \right] \right\}$ (3.36)
= $i \int d^4 y D_F(x-y) J(y)$.

Since

$$\left(\Box_x + m^2\right) D_F(x - y) = -i\delta^{(4)}(x - y), \qquad (3.37)$$

we arrive at

$$\left(\Box_x + m^2\right)\phi_c(x) = J(x), \qquad (3.38)$$

which means that $\phi_c(x)$ is a solution to the Klein-Gordon equation with source J(x). This is exactly what one would expect for a classical field coupled to J. Furthermore, inserting (3.30) and (3.36) into the effective action (3.34) it follows that

$$\Gamma[\phi_c] = \frac{i}{2} \int d^4x d^4y J(x) D_F(x-y) J(y) - \int d^4x J(x) \phi_c(x)$$

= $-\frac{1}{2} \int d^4x \phi_c(x) J(x) = -\frac{1}{2} \int d^4x \phi_c(x) \left(\Box_x + m^2\right) \phi_c(x)$ (3.39)
= $\int d^4x \left[\frac{1}{2} \left(\partial_\mu \phi_c(x)\right)^2 - \frac{1}{2} m^2 \phi_c^2(x)\right],$

where in the last step we have used integration by parts. In the free field case the effective action hence coincides with the classic action of the real scalar field.

For interacting theories, the generating functional can typically not be calculated exactly. Yet, one can evaluate the path integral (3.10) in the *saddle point approximation*. The solution ϕ_0 to the classical equations of motions (EOMs) is determined from

$$\frac{\delta S[\phi]}{\delta \phi(x)}\Big|_{\phi=\phi_0} = \frac{\delta S}{\delta \phi(x)}[\phi_0] = -J(x).$$
(3.40)

Then to leading order in the saddle point approximation one has

$$W[J] = \mathcal{N} \exp\left[iS[\phi_0] + i \int d^4x \, J(x)\phi_0(x)\right],$$

$$Z[J] = S[\phi_0] + \int d^4x \, J(x)\phi_0(x).$$
(3.41)

Comparing these results to (3.31) and (3.34) it is readily seen that

$$\phi_c(x) = \phi_0(x), \qquad \Gamma[\phi_c] = S[\phi_0].$$
 (3.42)

Hence to lowest order the effective action $\Gamma[\phi_c]$ is simply the classic action $S[\phi_0]$. Beyond the leading order the effective action will however receive quantum corrections and as a result one has generically $\Gamma[\phi_c] \neq S[\phi_0]$.

Let me add that the above formalism allows one to shed some light on a point that we glossed over in our discussion of spontaneous symmetry breaking in the script "Classical Field Theory". In this discussion we only talked about classic theories. However, one should ask whether the same or similar results would be obtained in the corresponding quantum theories. In fact, as it turns out spontaneous symmetry breaking should be analyzed with the effective action $\Gamma[\phi_c]$ (or more precisely the *effective potential*) rather than the classic action $S[\phi_0]$. If this is done, one can convince oneself that our discussion based on the classic theory makes sense even in the quantum theory. Yet, the classical analysis has to be viewed as a leading-order approximation. Since in weakly-coupled theory quantum corrections are always suppressed, the classic analysis of spontaneous symmetry breaking is therefore typically a good approximation to the full story.

3.5 Feynman Integrals from Path Integrals

In order to develop perturbation theory the path integral formalism, we split the Lagrangian as

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\text{int}} \,, \tag{3.43}$$

where \mathcal{L}_0 is the Lagrangian of the free theory, while \mathcal{L}_{int} contains all interactions. *E.g.*, in the case of ϕ^4 theory one has $\mathcal{L}_{int} = -\lambda/4! \phi^4$. The generating functional associated to \mathcal{L}_0 is called $W_0[J]$, while we will denote the full generating functional by W[J]. Explicitly, one has

$$W_{0}[J] = \mathcal{N}_{0} \int \mathcal{D}\phi \exp\left[i \int d^{4}x \left(\mathcal{L}_{0} + J(x)\phi(x)\right)\right],$$

$$W[J] = \mathcal{N} \int \mathcal{D}\phi \exp\left[i \int d^{4}x \left(\mathcal{L}_{0} + \mathcal{L}_{int} + J(x)\phi(x)\right)\right].$$
(3.44)

It follows that we can write W[J] as

$$W[J] = \mathcal{N} \exp\left[i\int d^4x \,\mathcal{L}_{\rm int}\left(-i\frac{\delta}{\delta J(x)}\right)\right] W_0[J]$$

$$= \mathcal{N}\left[1 + \sum_{n=1}^{\infty} \frac{i^n}{n!}\int d^4x_1 \dots d^4x_n \,\mathcal{L}_{\rm int}\left(-i\frac{\delta}{\delta J(x_1)}\right) \dots \mathcal{L}_{\rm int}\left(-i\frac{\delta}{\delta J(x_n)}\right)\right] W_0[J].$$
(3.45)

Here

$$\mathcal{N}^{-1} = \exp\left[i\int d^4x \,\mathcal{L}_{\rm int}\left(-i\frac{\delta}{\delta J(x)}\right)\right] W_0[J]\Big|_{J=0} \,, \tag{3.46}$$

to ensure that (3.12) is satisfied. The result (3.45) shows that W[J] is a perturbative series in terms of $W_0[J]$. But from (3.27) we know that

$$W_0[J] = \exp\left\{-\frac{1}{2}\int d^4x \, d^4y \, J(y)D_F(y-x)J(x)\right\},\qquad(3.47)$$

so in fact all functional derivatives can be carried out explicitly and lead to Feynman diagrams.

It is straightforward to see that the Green's functions (3.11) can be written as

$$\mathcal{G}^{(n)}(x_1,\ldots,x_n) = \mathcal{N} \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} \left[1 + \sum_{m=1}^{\infty} \frac{i^m}{m!} \int d^4 y_1 \ldots d^4 y_m \right] \times \mathcal{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(y_1)} \right) \cdots \mathcal{L}_{\text{int}} \left(-i \frac{\delta}{\delta J(y_m)} \right) W_0[J] \Big|_{J=0}^{(3.48)}$$

While this result looks kind of horrible it can in fact be worked out order by order in perturbation theory using *Wick's theorem*. This results in a sum over products of propagators D_F (suitably integrated) and each term can be associated to a Feynman graph. This is exactly what we have obtained before using canonical quantization, so the two approaches give at the end the same result. The path integral formalism is however more elegant.

3.6 A Simple Application

To become more familiar with the path integral formalism in QFT it seems worthwhile to consider a simple but educated example. In the following we will study a theory with a massless real scalar field described by the classic Lagrangian

$$\mathcal{L} = \frac{1}{2} \left(\partial_{\mu} \phi \right)^2 - \frac{\lambda}{3!} \phi^3 \,. \tag{3.49}$$

Remember that the dimension of the coupling λ is $[\lambda] = 1$.

In this case, $W_0[J]$ is given by (3.47) employing the massless Feynman propagator of a scalar field. The full generating functional takes the following form

$$W[J] = \mathcal{N} \exp\left[-i\int d^4x \,\frac{\lambda}{3!} \left(-i\frac{\delta}{\delta J(x)}\right)^3\right] W_0[J]$$

$$= \mathcal{N} \left\{ \sum_{V=0}^{\infty} \frac{1}{V!} \left[-\frac{i\lambda}{3!} \int d^4x \left(-i\frac{\delta}{\delta J(x)}\right)^3\right]^V \right\}$$

$$\times \left\{ \sum_{P=0}^{\infty} \frac{1}{P!} \left[-\frac{1}{2} \int d^4y d^4z \, J(y) D_F(y-z) J(z)\right]^P \right\},$$

(3.50)

where V and P count the number of vertices and propagators.

In order to evaluate this expression we can use the tools of Feynman diagrams. First, we determine the number of surviving sources which is equal to the number of external legs L of the graph. Since each propagator connects two points (external or internal) and our scalar theory (3.49) has only 3-point vertices, the number of external legs is given in terms of V and P by the simple formula

$$L = 2P - 3V. (3.51)$$

So instead of using V and P to order the perturbative series, we can also use V and L, and this is what we will do below.

For instance, let us consider V = L = 1. In this case we have

$$\begin{array}{c} & \\ \times & \\ x & y \end{array} = -\frac{\lambda}{2} \int d^4x d^4y J(x) D_F(x-y) D_F(y-y) \,, \qquad (3.52) \end{array}$$

where the dot in the diagram denotes a vertex while the cross indicates a source. The factor of 2 appearing in the denominator is the symmetry factor of the diagram (the ends of the line meeting at y can be interchanged without altering the result).

In the case of V = 2 and L = 0, we get on the other hand

$$x = \frac{\lambda^2}{12} \int d^4x d^4y D_F^3(x-y), \qquad (3.53)$$

and

$$\underbrace{ \sum_{x = y}}_{x = y} = \frac{\lambda^2}{8} \int d^4x d^4y D_F(x - x) D_F(x - y) D_F(y - y) .$$
 (3.54)

The value of the symmetry factor is in the first case $S = 2 \cdot 3! = 12$, where the factor of 2 arises from the exchange of x with y and the factor 3! stems from the possible ways to interchange the lines joining x and y. In the second case similar arguments lead to $S = 2 \cdot 2 \cdot 2 = 8$.

Dropping the multiplicative overall factors it is also easy to give a pictorial representation of the generating functional Z[J] as defined in (3.13). One has

$$Z[J] = \times \times + \cdots \qquad \mathcal{O}(\lambda^{0})$$

$$\times \longrightarrow + \times \longrightarrow + \cdots \qquad \mathcal{O}(\lambda)$$

$$(3.55)$$

$$(3.55)$$

$$\cdots$$

Notice that I have ordered the individual diagrams in the perturbative series of Z[J] corresponding to their number of external legs and powers of λ (or equivalent number of vertices).

Let us now try to calculate the classic field ϕ_c as defined in (3.31). Up to and including terms of order λ^2 , we get

$$\phi_{c}(x) = i \int d^{4}y \, D_{F}(x-y)J(y) + \frac{\lambda^{2}}{4} \int d^{4}y d^{4}z d^{4}u \, D_{F}(x-y)D_{F}^{2}(y-z)D_{F}(z-u)J(u)$$

$$= \underbrace{x \quad y}_{x \quad y} + \underbrace{x \quad y}_{z \quad u},$$
(3.56)

where the black square indicates the position x of the classic field $\phi_c(x)$. The factor of 1/4 in the $\mathcal{O}(\lambda^2)$ term arises again from the symmetry factor of the associated graph.

We now compute $\Box \phi_c(x)$. Recalling that D_F fulfils the Klein-Gordon equation (3.37), we obtain

$$\Box \phi_c(x) = J(x) - \frac{i\lambda^2}{4} \int d^4y d^4z D_F^2(x-y) D_F(y-z) J(z) , \qquad (3.57)$$

meaning that

$$J(x) = \Box \phi_c(x) + \frac{i\lambda^2}{4} \int d^4y d^4z D_F^2(x-y) D_F(y-z) J(z) \,. \tag{3.58}$$

We solve this equation recursively by making the ansatz

$$J(x) = J_0(x) + J_2(x)\lambda^2.$$
(3.59)

We get

$$J_0(x) = \Box \phi_c(x) , \qquad (3.60)$$

and

$$J_{2}(x) = \frac{i}{4} \int d^{4}y d^{4}z D_{F}^{2}(x-y) D_{F}(y-z) \Box_{z} \phi_{c}(z)$$

$$= \frac{i}{4} \int d^{4}y d^{4}z D_{F}^{2}(x-y) \left(\Box_{z} D_{F}(y-z) \right) \phi_{c}(z) \qquad (3.61)$$

$$= \frac{1}{4} \int d^{4}y D_{F}^{2}(x-y) \phi_{c}(y) .$$

To obtain the final result we have employed integration by parts twice, then used that D_F satisfies the Klein-Gordon equation with delta function source, which in turn allowed us to integrate over z. Combining (3.60) and (3.61), it follows that

$$J(x) = \Box \phi_c(x) + \frac{\lambda^2}{4} \int d^4 y \, D_F^2(x-y) \phi_c(y) \,. \tag{3.62}$$

We now move our attention to the effective action (3.34). At $\mathcal{O}(\lambda^0)$ one finds in terms of $J_0(x) = \Box \phi_c(x)$ the following expression

$$\Gamma_{0}[\phi_{c}] = \frac{i}{2} \int d^{4}x d^{4}y \,\Box \phi_{c}(x) D_{F}(x-y) \Box \phi_{c}(y) - \int d^{4}x \,(\Box \phi_{c}(x)) \,\phi_{c}(x)$$

$$= \int d^{4}x \,\frac{1}{2} \,(\partial_{\mu} \phi_{c}(x))^{2} \,.$$
(3.63)

Here I have again used integration by parts and the fact that D_F is a Green's function of the Klein-Gordon equation. This result looks quite familiar. In fact, we have already derived it in (3.39). It is the kinetic term of the classic action. There is of course no mass term, because we have set m = 0 by hand in the classic theory (3.49).

By inserting (3.62) into the definition (3.34), one can also show that the $\mathcal{O}(\lambda)$ part of the effective action takes the form

$$\Gamma_1[\phi_c] = \int d^4x \left[-\frac{\lambda}{3!} \phi_c^3(x) - \frac{\lambda}{2} D_F(0) \phi_c(x) \right] \,. \tag{3.64}$$

The term $-\lambda/3! \phi_c^3$ is again part of the classic action or Lagrangian (3.49). But the term

is a new contribution that arises from quantum corrections. These so-called *tadpole* contributions can be removed (by a proper renormalization) and therefore do not affect physical processes. So let's forget about them and press on.

At $\mathcal{O}(\lambda^2)$ one gets a contribution to $\Gamma[\phi_c]$ from the graph

$$\times \longrightarrow , \qquad (3.66)$$

i.e., the third diagram in the second line of Z[J] as given in (3.55) as well as a similar contribution from $-\int d^4x J(x)\phi_c(x)$. Using the expansion (3.62) of the source J, one finds after some algebra

$$\Gamma_2[\phi_c] = \int d^4x \left[-\frac{\lambda^2}{4} \phi_c(x) \int d^4y \, D_F^2(x-y) \phi_c(y) \right] = \quad \blacksquare \quad (3.67)$$

This is an interesting result. Since (3.67) is bilinear in ϕ_c the $\mathcal{O}(\lambda^2)$ term of the effective action corresponds to a *loop-induced mass term* for the classic field. So our scalar field will get a mass from radiative corrections even if we start with m = 0.

The general lesson to learn here is that if there is *no symmetry* that forbids a specific term in the Lagrangian, one better includes it in the theory. If one does not do this, one will always get it back in the quantum theory.