M.Phys Option in Theoretical Physics Hilary 2019: C6. Problem Sheet 5

1.) Consider a scalar field $\mathbf{\Phi}$ with a Lagrangian density

$$\mathcal{L} = (\partial_\mu \mathbf{\Phi})^\dagger (\partial^\mu \mathbf{\Phi}) - m^2 \mathbf{\Phi}^\dagger \mathbf{\Phi} - \lambda (\mathbf{\Phi}^\dagger \mathbf{\Phi})^2 ,$$

that possesses a global $SU(2)$ symmetry. Assume $m^2 < 0$ and $\lambda > 0$ in what follows.

a) Show that the $SU(2)$ Lie algebra consists of all complex, hermitian 2×2 matrices with vanishing trace.

b) Give a basis, τ^i, for the Lie algebra of $SU(2)$. What is the dimension of this Lie algebra? Calculate the associated *structure constants*.

(Hint: Given a set of generators T^a, the structure constants f^{abc} express the Lie brackets of pairs of generators as linear combinations of generators from the set, i.e. $[T^a, T^b] = i f^{abc} T^c$.)

c) Assume that the field $\mathbf{\Phi}$ transforms under the fundamental representation of $SU(2)$ and that it acquires a vacuum expectation value of the form $\langle \mathbf{\Phi} \rangle = (0, v)^T$ with $v \neq 0$. Compute v^2 in terms of m^2 and λ and find the sets of broken and unbroken $SU(2)$ generators. How many Goldstone modes do you expect for the symmetry breaking under consideration?

d) The global $SU(2)$ is now promoted to a gauge symmetry. Write down a minimal coupling between the gauge fields A^i_μ and the *fundamental* $SU(2)$ scalar $\mathbf{\Phi}$ and calculate the masses of the gauge bosons that result from this Higgs mechanism.

(Hint: For $SU(N)$ the original set of complex $N \times N$ matrices corresponds to the fundamental representation.)

e) Repeat the analysis you have performed in (d) now assuming that $\mathbf{\Phi}$ transforms under the *adjoint* representation, i.e. the representation the gauge fields A^i_μ belong to. Explain briefly similarities and differences between this model of electroweak symmetry breaking and the one actually present in the Standard Model of particle physics.

(Hint: In the adjoint representation the generators are given by $(T^a)^{bc} = -i f^{abc}$.)

2.) Consider a complex scalar field φ transforming as $\varphi(x) \to e^{i\alpha(x)} \varphi(x) = V(x)\varphi(x)$ under a local $U(1)$ symmetry. Any field theory should contain a kinetic term of the form $(\partial_\mu \varphi^*)(\partial^\mu \varphi)$, where the derivative of φ in the direction n^μ is defined by the limiting procedure

$$n^\mu \partial_\mu \varphi(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left[\varphi(x + \epsilon n) - \varphi(x) \right] .$$

a) Why is the above definition of the derivative impractical in the presence of the $U(1)$ gauge symmetry? Try to construct a sensible derivative, the *covariant derivative* D_μ, by introducing a *comparator* $U(y, x)$ with the following properties: $U(y, x)$ is identical to 1 for zero separation and required to be a pure phase $U(y, x) = \exp(i\phi(y, x))$. It transforms as

$$U(y, x) = V(y) U(y, x) V^*(x) ,$$

under the $U(1)$ symmetry.
b) By considering infinitesimal separated points find an explicit expression for D_μ involving a gauge *connection* (or field) $A_\mu(x)$. How do A_μ and $D_\mu \varphi$ transform under infinitesimal $U(1)$ transformations? Write down a gauge-invariant kinetic term for φ.
c) To complete the construction of a locally invariant Lagrangian, we must also find a

kinetic term for the gauge field A_μ. Such a term can, for example, be found by considering comparisons around a small square in space-time in the $(1,2)$ plane, namely

$$\mathbf{U}(x) = U(x, x+\epsilon\hat{2})U(x+\epsilon\hat{2}, x+\epsilon\hat{1}+\epsilon\hat{2})U(x+\epsilon\hat{1}+\epsilon\hat{2}, x+\epsilon\hat{1})U(x+\epsilon\hat{1}, x)\ .$$

Here $\hat{1}$ and $\hat{2}$ denote unit vectors in the 1- and 2-direction, respectively. How does $\mathbf{U}(x)$ transform under the $U(1)$ symmetry? By expanding $\mathbf{U}(x)$ to second power in ϵ, show that $\partial_1 A_2 - \partial_2 A_1$ or more generically

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\ ,$$

is locally gauge invariant. What is the gauge-invariant kinetic term associated to A_μ?

(Hint: To complete this step use that $U(x+\epsilon n, x) = \exp\left(-i\epsilon n^\mu A_\mu(x+\epsilon n/2) + \mathcal{O}(\epsilon^3)\right)$ after verifying that this result is correct.)

d) How does $[D_\mu, D_\nu]\varphi$ transform under the $U(1)$ symmetry? Find a connection between the commutator $[D_\mu, D_\nu]$ and the field-strength tensor $F_{\mu\nu}$.

e) Try to generalise the results of a), b), c) and d) to the case where the field φ transforms under a local SU(2) symmetry, that is, $\varphi(x) \to e^{i\alpha^i(x)\tau_i}\varphi(x) = V(x)\varphi(x)$ with $\tau_i = \sigma_i/2$ and σ_i the usual Pauli matrices ($i = 1,2,3$). Comment on similarities and differences in the derivations and final results.

3.) Consider three real, free scalar fields ϕ_a with the same mass m and Lagrangian density $\mathcal{L} = \frac{1}{2}\sum_{a=1}^{3}\left(\partial_\mu\phi_a\partial^\mu\phi_a - m^2\phi_a^2\right)$.
a) Show that this theory has an $SO(3)$ symmetry and derive the Noether currents and charges associated to this symmetry.
b) Generalising the results for a single real scalar field,

$$\phi(x) = \int d^3\tilde{k}\,\left(a(k)e^{-ikx} + a^\dagger(k)e^{ikx}\right)\ ,$$
$$\pi(x) = -i\int d^3\tilde{k}\,\omega_{\mathbf{k}}\left(a(k)e^{-ikx} - a^\dagger(k)e^{ikx}\right)\ ,$$

quantise this theory using the equal-time commutation relations

$$[\phi(t,\mathbf{x}), \phi(t,\mathbf{y})] = [\pi(t,\mathbf{x}), \pi(t,\mathbf{y})] = 0\ , \quad [\pi(t,\mathbf{x}), \phi(t,\mathbf{y})] = -i\delta^{(3)}(\mathbf{x}-\mathbf{y})\ .$$

Here $d^3\tilde{k} = d^3k/(2\pi)^3 1/(2\omega_{\mathbf{k}})$ with $\omega_{\mathbf{k}} = \sqrt{\mathbf{k}^2 + m^2}$, $k_\mu = (\omega_{\mathbf{k}}, \mathbf{k})$, and $a(k)$ ($a^\dagger(k)$) annihilate (create) single particle states with momentum k. Find the conjugate momenta, write down canonical commutation relations, expand the fields in terms of creation and annihilation operators, work out the commutation relations for $a_a(k)$ and $a_a^\dagger(k)$ and use the creation operators to construct the Fock space.
c) Find expressions for the conserved Noether charges in terms of creation and annihilation operators and use these expressions to verify that the Noether charges form an $SO(3)$ algebra.
d) Compute the action of the Noether charges on one-particle states and show that the one-particle states with a given four-momentum k form a representation of $SO(3)$. What is the "spin" of this representation?

4.) Let's be adventurous and consider a field theory in *two dimensions*, with coordinates $(\sigma_\mu) = (\tau, \sigma)$, where $\mu, \nu, \dots = 0, 1$ and a collection of real scalar fields $X^a = X^a(\tau, \sigma)$, where $a, b, \dots = 0, ..., n-1$. The Lagrangian density is given by $\mathcal{L} = -\frac{1}{2} g_{ab} \partial_\mu X^a \partial^\mu X^b$ with associated action $S = \int d^2\sigma\, \mathcal{L}$. Two-dimensional indices $\mu, \nu, \dots$ are lowered and raised with the two-dimensional Minkowski metric $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1)$ and its inverse. We take the metric g_{ab} to be the Minkowski metric in n dimensions, that is, $(g_{ab}) = \mathrm{diag}(-1, 1, \dots, 1)$. Also, we assume that the fields X^a are periodic with period $l = 2\pi$ in the spatial coordinate σ, so that $X^a(\tau, \sigma) = X^a(\tau, \sigma + l)$.
a) Derive the equations of motion for X^a using the variational principle.
b) Introduce "light-cone" coordinates $\sigma^\pm = \tau \pm \sigma$ and show that the equations of motion can be written in the form

$$\frac{\partial^2 X^a}{\partial \sigma^+ \partial \sigma^-} = 0 \; .$$

c) Show that the most general solution to this equation can be written as $X^a(\tau, \sigma) = X^a_-(\sigma^-) + X^a_+(\sigma^+)$ with

$$X^a_\pm(\sigma^\pm) = \frac{1}{2} x^a + p^a \sigma^\pm + \sum_{n \neq 0} \alpha^a_{\pm n} e^{-in\sigma^\pm} \; ,$$

where x^a, p^a are real constants and $\alpha^a_{\pm n}$ are complex constants with $\alpha^a_{\pm -n} = (\alpha^a_{\pm n})^*$.
d) Determine the conjugate momenta for X^a and quantise the theory by imposing canonical commutation relations. Find the commutation relations for the operators x^a, p^a and $\alpha^a_{\pm n}$ in the above expansion such that the canonical commutation relations are satisfied.
e) Show that $\alpha^a_{\pm n}$ with $n < 0$ ($n > 0$) can be interpreted as creation (annihilation) operators and construct the Fock space. Then verify that the states $\alpha^0_{\pm n}|0\rangle$ for $n < 0$ have negative norm. Speculate on what might have gone wrong.

5.) Consider a real scalar $\phi(t, x)$ field living on a two-dimensional space-time and defined on an interval $x \in [0, L]$ with *Dirichlet boundary conditions* $\phi(t, 0) = \phi(t, L) = 0$.
a) Show that the (classical) positive- and negative-frequency solutions to the Klein-Gordon equation that also satisfy the boundary conditions have the form

$$\phi^{(\pm)}_n(t, x) = \frac{1}{\sqrt{\omega_n L}}\, e^{\pm i\omega_n t}\, \sin(k_n x) \; .$$

Give the expression for k_n in terms of L. How is ω_n related to k_n?
b) Now quantise the field $\phi(t, x)$, keeping in mind that momentum is discrete

$$\phi(t, x) = \sum_{n=1}^{\infty} \left(\phi^{(-)}_n(t, x)\, a_n + \phi^{(+)}_n(t, x)\, a^\dagger_n \right) \; ,$$

with the annihilation/creation operators satisfying $[a_n, a_m] = [a^\dagger_n, a^\dagger_m] = 0$ and $[a_n, a^\dagger_m] = \delta_{mn}$. Compute the vacuum expectation value $\langle 0|\mathcal{H}|0\rangle$ of the Hamiltonian density

$$\mathcal{H} = \frac{1}{2}\left[\dot{\phi}^2 + (\partial_x \phi)^2 + m^2 \phi^2 \right] \; .$$

Integrating your result over the interval $[0, L]$ and show that the total vacuum energy is

$$E_0(L) = \frac{1}{2} \sum_{n=1}^{\infty} \omega_n \; .$$

c) Since this quantity is infinite, we need some form of regularisation in order to handle the divergence. Let us introduce an exponentially damping function $\exp(-\delta\omega_n)$ with $\delta > 0$ in the sum, and consider for simplicity the case of a massless field. Prove that in this case the vacuum energy can be written as

$$E_0(L,\delta) = \frac{\pi}{8L} \sinh^{-2}\left(\frac{\delta\pi}{2L}\right) .$$

Take the limit $\delta \to 0$ and determine the vacuum energy for the case when no boundary conditions are imposed. With all this at hand calculate the *Casimir force*, that is, the attractive force associated to the mismatch between the vacuum energy of the unbounded space and that of the theory on the interval.