Quantum Mechanics HT 2025: Problem Sheet 5

Transformations

5.1 Reflection symmetry around a point $\mathbf{x}_0$

Let $P_{\mathbf{x_0}}$ be the operator that induces reflections around a point $\mathbf{x_0}$. Argue that

$$\begin{aligned}\hat{P}_{\mathbf{x_0}}|\mathbf{x_0}+\mathbf{x}\rangle &= |\mathbf{x_0}-\mathbf{x}\rangle\ ,\\ \hat{P}_{\mathbf{x_0}}\hat{\mathbf{x}}\hat{P}_{\mathbf{x_0}} &= 2\mathbf{x_0}\mathbf{1}-\hat{\mathbf{x}}\ ,\\ \hat{P}_{\mathbf{x_0}}\hat{\mathbf{p}}\hat{P}_{\mathbf{x_0}} &= -\hat{\mathbf{p}},\end{aligned}$$

and that the transformed wave function fulfils

$$\psi'(\mathbf{x}) = \psi(2\mathbf{x_0}-\mathbf{x}).$$

5.2 For which potentials V is the Hamiltonian $H = \frac{\hat{\boldsymbol{p}}^2}{2m} + V(\hat{\mathbf{x}})$ translationally invariant?

5.3 Show that the orbital angular momentum operators $\hat{L}_a$ $(a = x, y, z)$ are Hermitian.

5.4 A spinless QM system is called *rotationally invariant* if its Hamiltonian commutes with the orbital angular momentum operators $[H, \hat{L}_a] = 0$, $a = x, y, z$. Rotational invariance expresses the fact the energy measurements remain unchanged under rotations of the system. If the Hamiltonian commutes only with $\hat{L}_z$ it is called invariant under rotations around the z-axis. Consider Hamiltonians of the form

$$H = \frac{\hat{\mathbf{p}}^2}{2m} + V(\hat{\mathbf{x}}).$$

Show that potentials that depend only on the distance $\|\mathbf{x}\|$ lead to rotationally symmetric Hamiltonians, while potentials that depend on x and y only through the combination $x^2 + y^2$ leads to Hamiltonians that invariant under rotations around the z-axis.

5.5 Show that

$$\lim_{N\to\infty}\left[1+\frac{x}{N}\right]^N = e^x.$$

Give arguments that an analogous formula holds for operators.

Orbital angular momentum

5.6

(a) Show by explicit calculation using $\hat{L}_i = \epsilon_{ijk}\hat{x}_j\hat{p}_k$ that $[\hat{L}_i, \hat{x}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{x}_k$ and $[\hat{L}_i, \hat{p}_j] = \mathrm{i}\hbar\epsilon_{ijk}\hat{p}_k$.

(b) Evaluate $[\hat{L}_x, \hat{L}_y]$ by writing $\hat{L}_y = \hat{z}\hat{p}_x - \hat{x}\hat{p}_z$ and using the results from part (a) of this question.

(c) Show that in the position representation we have

$$\langle\mathbf{x}|\hat{L}_i|\psi\rangle = \hat{\mathrm{L}}_i\psi(\mathbf{x})\ ,$$

and obtain explicit expressions for the differential operators $\hat{\mathrm{L}}_i$. Show that for any differentiable function f

$$(\hat{\mathrm{L}}_x\hat{\mathrm{L}}_y - \hat{\mathrm{L}}_y\hat{\mathrm{L}}_x)f(x,y,z) = \mathrm{i}\hbar\hat{\mathrm{L}}_z f(x,y,z).$$

Since this holds for any f it can be written as an operator equation $[\hat{\mathrm{L}}_x, \hat{\mathrm{L}}_y] = \hat{\mathrm{L}}_x\hat{\mathrm{L}}_y - \hat{\mathrm{L}}_y\hat{\mathrm{L}}_x = \mathrm{i}\hbar\hat{\mathrm{L}}_z$, as you will have found in part (b). Deduce similar expressions for $[\hat{\mathrm{L}}_y, \hat{\mathrm{L}}_z]$ and $[\hat{\mathrm{L}}_z, \hat{\mathrm{L}}_x]$.

(d) Defining $\hat{\boldsymbol{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, show that (Hint: remember $[\hat{A}, \hat{B}\hat{C}] = \hat{B}[\hat{A}, \hat{C}] + [\hat{A}, \hat{B}]\hat{C}$)

$$[\hat{L}_x, \hat{\boldsymbol{L}}^2] = [\hat{L}_y, \hat{\boldsymbol{L}}^2] = [\hat{L}_z, \hat{\boldsymbol{L}}^2] = 0.$$

(e) Show that in spherical polar co-ordinates the differential operators $\hat{\mathrm{L}}_i$ take the form

$$\hat{\mathrm{L}}_x = i\sin\phi\frac{\partial}{\partial\theta} + i\cot\theta\cos\phi\frac{\partial}{\partial\phi}\ , \quad \hat{\mathrm{L}}_y = -i\cos\phi\frac{\partial}{\partial\theta} + i\cot\theta\sin\phi\frac{\partial}{\partial\phi}\ , \quad \hat{\mathrm{L}}_z = -i\frac{\partial}{\partial\phi}\ .$$

5.7 (a) Verify that the three functions $\cos\theta$, $\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}$ and $\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}$ are all eigenfunctions of $\hat{\boldsymbol{L}}^2$ and $\hat{\mathrm{L}}_z$.
(b) Find normalization constants N for each of the above functions so that

$$\int_0^{2\pi} \mathrm{d}\phi \int_0^{\pi} \mathrm{d}\theta\, \sin\theta\, N^2\, |\psi(\theta,\phi)|^2 = 1.$$

(c) Once normalized, these functions are called spherical harmonics and given the symbol $Y_\ell^m(\theta,\phi)$. Hence deduce that your results are consistent with the functions:

$$Y_1^0(\theta,\phi) = \sqrt{\frac{3}{4\pi}}\cos\theta; \qquad Y_1^1(\theta,\phi) = -\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\mathrm{i}\phi}; \qquad Y_1^{-1}(\theta,\phi) = \sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}.$$

(Note, you can't use this method to get the signs of Y_1^0, Y_1^1 and Y_1^{-1}. The minus sign in Y_1^1 can be deduced by using a raising operator $\hat{L}_+$ on Y_1^0. This is not required.)
(d) Rewrite these functions in terms of Cartesian variables [$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$]. Sketch $|Y_1^0|^2$, $|Y_1^1|^2$ and $|Y_1^{-1}|^2$. (They are angular functions, so keep r fixed and look only at the angle dependence. A cross section in the x–z plane will do. Why?)

5.8 The angular part of a system's wavefunction is

$$\langle\theta,\phi|\psi\rangle \propto (\sqrt{2}\cos\theta + \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi} - \sin\theta\,\mathrm{e}^{\mathrm{i}\phi}).$$

What are the possible results of measurement of (a) $\hat{\boldsymbol{L}}^2$, and (b) $\hat{L}_z$, and their probabilities? What is the expectation value of $\hat{L}_z$?

5.9 A system's wavefunction is proportional to $\sin^2\theta\,\mathrm{e}^{2\mathrm{i}\phi}$. What are the possible results of measurements of (a) $\hat{L}_z$ and (b) $\hat{\boldsymbol{L}}^2$?

5.10 A system's wavefunction is proportional to $\sin^2\theta$. What are the possible results of measurements of (a) $\hat{L}_z$ and (b) $\hat{\boldsymbol{L}}^2$? Give the probabilities of each possible outcome.

5.11 A particle of mass m is described the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\nabla^2 - \frac{e^2}{4\pi\epsilon_0 r} - e\mathcal{E}\hat{x}.$$

(a) What is the physical origin of the last term in $\hat{H}$?
(b) Which of the observables represented by the operators $\hat{\boldsymbol{L}}^2$, $\hat{L}_x$, $\hat{L}_y$ and $\hat{L}_z$ are constants of the motion assuming (i) $\mathcal{E} = 0$; (ii) $\mathcal{E} \neq 0$ (Hint: Use the results for $[\hat{L}_i, \hat{x}]$ from [5.6].)

5.12 Show that $\hat{L}_i$ commutes with $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{p}}$, $\hat{\boldsymbol{x}}\cdot\hat{\boldsymbol{x}}$ and $\hat{\boldsymbol{p}}\cdot\hat{\boldsymbol{p}}$.
Evaluate (think carefully!) the commutator $\left[L_x, \left((\hat{\boldsymbol{x}}\times\hat{\boldsymbol{p}})\cdot\left(\hat{\boldsymbol{L}}\times\hat{\boldsymbol{p}}\right)\right)^2\right]$.