Quantum Mechanics HT 2025: Problem Sheet 4

The simple harmonic oscillator

4.1 After choosing units in which everything, including $\hbar = 1$, the Hamiltonian of a harmonic oscillator may be written $\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2)$, where $[\hat{x}, \hat{p}] = \mathrm{i}$. Show that if $|\psi\rangle$ is a ket that satisfies $H|\psi\rangle = E|\psi\rangle$, then

$$\frac{1}{2}(\hat{p}^2 + \hat{x}^2)(\hat{x} \mp \mathrm{i}\hat{p})|\psi\rangle = (E \pm 1)(\hat{x} \mp \mathrm{i}\hat{p})|\psi\rangle.$$

Explain how this algebra enables one to determine the energy eigenvalues of a harmonic oscillator.

For more general units, show that the ladder operators can be written

$$a = \frac{m\omega\hat{x} + i\hat{p}}{\sqrt{2m\hbar\omega}} \quad \text{and} \quad a^\dagger = \frac{m\omega\hat{x} - i\hat{p}}{\sqrt{2m\hbar\omega}}.$$

4.2 Given that $\hat{a}|n\rangle = \alpha|n-1\rangle$ and $E_n = (n + \frac{1}{2})\hbar\omega$, where the annihilation operator of the harmonic oscillator is

$$\hat{a} \equiv \frac{m\omega\hat{x} + \mathrm{i}\hat{p}}{\sqrt{2m\hbar\omega}},$$

show that $\alpha = \sqrt{n}$. Hint: consider $|\hat{a}|n\rangle|^2$.

4.3 The pendulum of a grandfather clock has a period of 1 s and makes excursions of 3 cm either side of dead centre. Given that the bob weighs 0.2 kg, around what value of n would you expect its non-negligible quantum amplitudes to cluster?

4.4 Show that the minimum value of $E(p, x) \equiv p^2/2m + \frac{1}{2}m\omega^2 x^2$ with respect to the real numbers p, x when they are constrained to satisfy $xp = \frac{1}{2}\hbar$, is $\frac{1}{2}\hbar\omega$. Explain the physical significance of this result.

4.5 How many nodes are there in the wavefunction $\langle x|n\rangle$ of the nth excited state of a harmonic oscillator?

4.6 Show that for a harmonic oscillator that wavefunction of the second excited state is $\langle x|2\rangle = \text{constant} \times (x^2/\ell^2 - 1)\mathrm{e}^{-x^2/4\ell^2}$, where $\ell \equiv \sqrt{\hbar/2m\omega}$ and find the normalising constant.

4.7 Use

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a} + \hat{a}^\dagger) = \ell(\hat{a} + \hat{a}^\dagger)$$

to show for a harmonic oscillator that in the energy representation the operator $\hat{x}$ is

$$\hat{x}_{jk} = \ell \begin{pmatrix}
0 & \sqrt{1} & 0 & 0 & \cdots & & & & \\
\sqrt{1} & 0 & \sqrt{2} & 0 & & & & & \\
0 & \sqrt{2} & 0 & \sqrt{3} & \cdots & & & & \\
 & & \sqrt{3} & \cdots & & & & & \\
\cdots & & \cdots & & \cdots & & \cdots & & \\
 & & & \cdots & 0 & \sqrt{n-1} & \cdots & & \\
 & & & & \sqrt{n-1} & 0 & \sqrt{n} & & \\
 & & & & & \sqrt{n} & 0 & \sqrt{n+1} & \cdots \\
 & & & & & & \sqrt{n+1} & 0 & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{pmatrix}$$

Calculate the same entries for the matrix $\hat{p}_{jk}$.

4.8 At $t = 0$ the state of a harmonic oscillator of mass m and frequency ω is

$$|\psi\rangle = \frac{1}{2}|N-1\rangle + \frac{1}{\sqrt{2}}|N\rangle + \frac{1}{2}|N+1\rangle.$$

Calculate the expectation value of x as a function of time and interpret your result physically in as much detail as you can.