9. $\underline{L} = \underline{r} \times \underline{p}$

In classical mechanics $\underline{r}$, $\underline{p}$ are vectors

In quantum " $\underline{r}$, $\underline{p}$ are operators; so for example $\underline{L}$ can be written as a differential operator

$$\underline{L} = -i\hbar\, \underline{r} \times \underline{\nabla}$$

Compute $L_z f(r)$

$$= -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) f\left(\sqrt{x^2+y^2+z^2}\right).$$

$$= -i\hbar \frac{(xy - yx)}{\sqrt{x^2+y^2+z^2}} f'\left(\sqrt{x^2+y^2+z^2}\right)$$

$$= 0$$

similarly for L_x and L_y.

a) Compute $L_z(z\, f(r)) = (L_z z)\, f(r) + z\, L_z f(r).$

$= 0$ so L_z definitely zero

b) $L_x(z\, f(r)) = (L_x z)\, f(r) + z\, L_x f(r).$

$$= -i\hbar \left(\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right) z \right) f(r) + 0$$

$= -i\hbar\, y\, f(r)$ so not an eigenstate of L_x $\therefore$ doesn't have a definite value of L_x.

c) $\langle L_z \rangle = \int \psi^* L_z \psi \, d^3\underline{r}$

$= 0$

$\langle L_x \rangle = \int \psi^* L_x \psi \, d^3\underline{r}$

$= \int f^*(r) \, z \, (-i\hbar y) \, f(r) \, d^3\underline{r}$

$= 0$ (zy is odd function)

$$L_x \frac{1}{\sqrt{2}} (y + iz) f(r) = \left[-i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)(y + iz) \right] f(r).$$

$$= -i\hbar (iy - z) f(r).$$

$$= \hbar (y + iz) f(r)$$

so it is an eigenfn with eigenvalue $+\hbar$.

$$L_x \frac{1}{\sqrt{2}} (y - iz) f(r) = \left[-i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)(y - iz) \right] f(r).$$

$$= -i\hbar (-iy - z) f(r).$$

$$= -\hbar (y - iz) f(r)$$

so it is an eigenfn with eigenvalue $-\hbar$

Note that $|\psi\rangle = \frac{1}{i\sqrt{2}} \left(|L_x = +1\rangle - |L_x = -1\rangle \right) = |L_z = 0\rangle$

d) Measure L_z : get 0 state remains $|L_z = 0\rangle$

Measure L_x : get $+1$ with prob $\frac{1}{2}$, -1 with prob $\frac{1}{2}$

e) Measure L_x: get $+1$, prob $\frac{1}{2}$, state becomes $|L_x=+1\rangle$

get -1, prob $\frac{1}{2}$, state becomes $|L_x=-1\rangle$

$$|L_x=+1\rangle \leftrightarrow \frac{1}{\sqrt{2}}(y+iz)\,f(r)$$

eigenstates of L_z are

$z\,f(r)$	0
$\frac{1}{\sqrt{2}}(x+iy)\,f(r)$	$+1$
$\frac{1}{\sqrt{2}}(x-iy)\,f(r)$	$-1.$

so $$|L_x=+1\rangle = \frac{i}{\sqrt{2}}|L_z=0\rangle + \frac{1}{2i}\left(|L_z=+1\rangle - |L_z=-1\rangle\right).$$

a measurement of L_z gives

$+1$	prob $1/4$
0	" $1/2$
-1	" $1/4$

Clearly it works the same way with $|L_x=-1\rangle$ so these are the possible outcomes for the L_z measurement.