# QFT H.T. 2010

John Wheater

## Problems 1

To be handed in by the end of Friday of 4th week.

1. Draw all the tree-graph Feynaman diagrams for the process $e^-e^- \to e^-e^-\gamma$ and write down the corresponding expression for the matrix element. Repeat the exercise for $e^-e^- \to e^-e^-\gamma\gamma$; show that the matrix element is gauge invariant in each case.

2. The unitary matrix $C$ is defined by $C\gamma_\mu^T C^{-1} = -\gamma_\mu$ where "$T$" denotes transpose. Consider the transformation on the Dirac field $\psi \to C\overline{\psi}^T$:

   (a) Show that $\overline{\psi}\psi \to -\psi^T\overline{\psi}^T$ under the transformation and hence that $\overline{\psi}\psi$ is invariant.

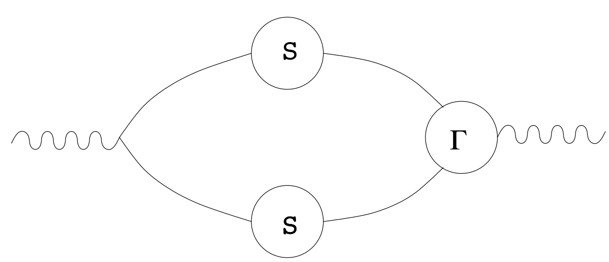
   (b) Show that $\overline{\psi}\gamma_\mu\psi \to \psi^T\gamma_\mu^T\overline{\psi}^T$.

   (c) Hence show that the Dirac action is invariant under $\psi \to C\overline{\psi}^T, \quad A_\mu \to -A_\mu$.

   (d) Show that the rest of the action and the path integral measure are also invariant under these transformations and hence prove that Green's functions with no external fermion lines and an odd number of external photon lines vanish. Give a physical explanation of this.

3. The one-particle-irreducible photon self energy is given by the diagram

   $\Pi_{\mu\nu}(k) =$




   where $\tilde{\Gamma}^\mu$ represents the full one particle irreducible vertex function and $S_F$ the full fermion two point function. By considering possible diagrams contributing to $\Pi_{\mu\nu}$ explain why this is so. Using the identity

$$k_\mu\tilde{\Gamma}^\mu(k, p, -p-k) = eS_F(p)^{-1} - eS_F(p+k)^{-1}$$

prove that $k^\mu \Pi_{\mu\nu}(k) = 0$.

Use this Ward Identity for the photon self-energy to prove that the full photon two point function $D_{\mu\nu}(k)$ has a pole only at $k^2 = 0$. What is the physical significance of this result?

4. By considering the BRST variation of $\eta(x) A_\lambda(y) A_\sigma(z) A_\rho(u)$ derive a Ward identity for the one-particle-irreducible photon four-point function. Hence show that the one loop contribution to the four-point function is ultra-violet finite. Why is this crucial to the renormalizability of QED?

5. Define the action of the operator $Q$ by the BRST-like properties

$$
\begin{aligned}
QA_\mu &= \partial_\mu \omega \\
Q\,\eta &= \xi^{-1} \partial_\mu A^\mu \\
Q\,\psi &= ie\omega\psi \\
Q\,\overline{\psi} &= ie\omega\overline{\psi} \\
Q\,\omega &= 0
\end{aligned}
$$

Show that $Q^2 X = 0$ for any field $X$ up to the equations of motion (ie you have to use the classical equations of motion following from the gauge-fixed QED Lagrangian). Now show that by introducing an extra field (call it $F$) and modifying the actions of $Q$ it can be arranged that $Q^2 X = 0$ without recourse to the classical equations of motion. If $F$ is included how does the gauge-fixed QED Lagrangian have to be modified to make it invariant under the BRST transformations

$$X \to X + \epsilon Q X\,? \tag{1}$$