Vectors and Matrices, Problem Set 4

Eigenvectors, eigenvalues and diagonalization

Prof Andre Lukas, MT2013

Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Find the eigenvalues and a set of normalized eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} .$$

Verify that its eigenvectors are mutually orthogonal and construct an orthogonal matrix R such that $R^T M R$ is diagonal.

2. Find the eigenvalues and normalized eigenvectors of the Hermitian matrix

$$H = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix} .$$

and construct a unitary matrix U such that $U^\dagger H U$ is diagonal. (Remember to use the standard *hermitian* scalar product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^\dagger \mathbf{w}$.)

3. A curve in two-dimensional space is defined by all $\mathbf{x} = (x, y)^T$ which solve the equation $x^2 + 3y^2 - 2xy = 1$.

(a) Show that this equation can be written in the form $\mathbf{x}^T A \mathbf{x} = 1$ and determine the matrix A.

(b) By diagonalizing the matrix A, show that this curve is an ellipse and determine the length of its two axis.

4. (a) Diagonalize the two-dimensional rotation

$$R = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

over the complex numbers.

(b) Why can the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

not be diagonalized?

5. A real, symmetric $n \times n$ matrix M is called positive definite if $\mathbf{v}^T M \mathbf{v} > 0$ for all vectors $\mathbf{v}$, negative definite if $-M$ is positive definite and indefinite otherwise.

(a) Show that M is positive definite iff all its eigenvalues are positive and that it is negative definite iff all its eigenvalues are negative.

(b) Focus on the two-dimensional case, $n = 2$, and denote the two eigenvalues of M by λ_1 and λ_2. Why is $\mathrm{tr}(M) = \lambda_1 + \lambda_2$ and $\det(M) = \lambda_1 \lambda_2$?

(c) For $n = 2$, formulate criteria for positive definiteness/negative definiteness/indefiniteness of M in terms of $\mathrm{tr}(M)$ and $\det(M)$.

(d) Are the following matrices positive definite, negative definite or indefinite?

$$M_1 = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 2 & 3 \\ 3 & -2 \end{pmatrix}, \quad M_3 = \begin{pmatrix} -4 & 3 \\ 3 & -5 \end{pmatrix}.$$

6. (a) Show that the characteristic polynomial of an arbitrary 2×2 matrix A can be written as

$$\chi_A(\lambda) = \lambda^2 - \mathrm{tr}(A)\lambda + \det(A) .$$

(b) Do the same for an arbitrary 3×3 matrix A and show that

$$\chi_A(\lambda) = -\lambda^3 + \mathrm{tr}(A)\lambda^2 - \frac{1}{2}(\mathrm{tr}(A)^2 - \mathrm{tr}(A^2))\lambda + \det(A) .$$

(c)* For a general $n \times n$ matrix A, what are the coefficients of λ^n, λ^{n-1} and the constant term in the characteristic polynomial $\chi_A(\lambda)$.

7. Consider real, symmetric $n \times n$ matrices P which satisfy $P^2 = P$. (Such matrices are also referred to as projectors.)

(a) Show that the possible eigenvalues of P are either 0 or 1. What does the diagonalized form of P look like?

(b) Explain why such matrices are called projectors and why the dimension of the space projected onto is given by $\mathrm{tr}(P)$.

(c) For an n-dimensional unit vector $\mathbf{n}$ with components n_i let Q be the matrix with components $Q_{ij} = n_i n_j$. Show that Q is a projector and that it projects to a one-dimensional space.

(d) Show that the matrix $P = \mathbf{1} - Q$ with Q defined as in part (c) is also a projector. What is the dimension of the space it projects onto?

8.* (a) Unitary matrices U are characterized by $U^\dagger U = \mathbf{1}$. Show that eigenvalues λ of unitary matrices must be phases, that is, $|\lambda| = 1$.

(b) Consider three-dimensional rotations R. Intuitively, it is clear that R has an eigenvector, $\mathbf{n}$, with eigenvalue 1, which corresponds to the axis of rotation. Construct an argument which proves this fact, that is, show that a three-dimensional real matrix R with $R^T R = \mathbf{1}$ and $\det(R) = 1$ always has an eigenvalue 1.

(c) Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$ be an ortho-normal basis of $\mathbb{R}^3$, where $\mathbf{n}$ is the eigenvector with eigenvalue 1 of a rotation R. Work out the matrix which represents the rotation in this basis.

(d) Show that the angle of rotation, φ, of a three-dimensional rotation R satisfies $\cos\varphi = (\mathrm{tr}(R) - 1)/2$.

(e) Show that the matrix

$$R = \frac{1}{3\sqrt{2}} \begin{pmatrix} 3 & 0 & 3 \\ -1 & -4 & 1 \\ 2\sqrt{2} & -\sqrt{2} & -2\sqrt{2} \end{pmatrix}$$

is a rotation. Compute the characteristic polynomial of R and verify that 1 is an eigenvalue. Compute the axis of rotation, $\mathbf{n}$, and $\cos\varphi$, where φ is the angle of rotation.

9.* Consider the system of second order differential equations

$$\frac{d^2\mathbf{x}}{dt^2} = -M\mathbf{x}\ ,$$

where

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}\ , \quad \frac{d^2\mathbf{x}}{dt^2} = \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \vdots \\ \frac{d^2x_n}{dt^2} \end{pmatrix}\ ,$$

and M is a real, symmetric matrix.

(a) Explain how this system of differential equations can be solved by diagonalizing M.

(b) Discuss the relation between the eigenvalues of M and the qualitative behaviour of the solutions.

(c) Focus on the two-dimensional system

$$\begin{aligned} \frac{d^2x_1}{dt^2} &= -kx_1 - lx_2 \\ \frac{d^2x_2}{dt^2} &= -lx_1 - kx_2 \end{aligned}$$

where k and l are arbitrary real constants. Depending on the values of k and l, discuss the qualitative behaviour of the solutions of this system.

Additional computational problems

Computational methods, both numerical and symbolic, are of increasing importance in physics and symbolic computational tools have become significantly more powerful over the past decade or so. This is changing the way physicists work. Much as the introduction of the pocket calculator some 50 years ago has made by-hand numerical calculations unnecessary, modern systems such as Mathematica, can now take over standard symbolic calculations, such as algebraic manipulations or integration. This facilitates powerful checks of by-hand calculations but also allows for calculations which are virtually intractable with a pen-and-paper approach. The following problems present an opportunity to practice some of these methods in the context of topics from Linear Algebra. They are supplementary and voluntary but strongly recommended and hopefully a fun way to engage with symbolic computations early on. The problems are meant for realisation in Mathematica which can be downloaded from the university server. Mathematica is easy to use, has good built-in documentation and many high-level mathematical functions - you can start to experiment immediately.

C1) (Eigenvalues and eigenvectors) Check some of the results for eigenvalues and eigenvectors from the problem set. Specifically,

(a) Check the results from question 1.

(b) Check the results from question 2.

(c) Check the results from question 3 and visualise the results by a suitable plot.

C2) (Characteristic polynomial)
Analyze the characteristic polynomial computationally.

(a) Proof the general form of the characteristic polynomial for 2×2 matrices given in question 6 a).

(b) Proof the general form of the characteristic polynomial for 3×3 matrices given in question 6 b).

(c) Find a nice form of the characteristic polynomial in terms of the determinant and traces for general 4×4 matrices.

C3) (Second order differential equations)
Write a short piece of code which solves differential equations of the type given in question 9 via diagonalisation. Allow for options to built in initial conditions $\mathbf{x}_0 = \mathbf{x}(0)$ and $\mathbf{v}_0 = \dot{\mathbf{x}}(0)$. Try the code on examples and visualise the solutions by plotting.