Vectors and Matrices, Problem Set 3

Scalar products and determinants

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Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Calculate the determinants of the matrices

$$(a)\ A = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}\,,\quad (b)\ B = \frac{1}{\sqrt{8}}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}\,.$$

Are the matrices (i) real, (ii) diagonal, (iii) symmetric, (iv) antisymmetric, (v) singular, (vi) orthogonal, (vii) Hermitian, (viii) anti-Hermitian, (ix) unitary?

2. Use the Gram-Schmidt procedure to find an ortho-normal basis of $\mathbb{R}^3$ (with the standard scalar product), starting with the basis

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\,,\quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}\,,\quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}\,.$$

Check your result.

3. Let A be an $n \times n$ matrix.

(a) Why are the statements "A is invertible" and $\det(A) \neq 0$ equivalent?

(b) For which values of the parameters a, b is the matrix

$$A = \begin{pmatrix} a & 1 & a \\ 1 & b & -1 \\ 0 & -1 & a \end{pmatrix}$$

not invertible?

4.* In $\mathbb{R}^n$, we have $n-1$ linearly independent vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n-1}$. Define the vector $\mathbf{w}$ with components $w_i = \det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{e}_i)$, where $\mathbf{e}_i$ are the standard unit vectors.

(a) Show that $\mathbf{w}$ is perpendicular (with respect to the standard scalar product in $\mathbb{R}^n$) to all vectors $\mathbf{v}_a$, where $a = 1, \dots, n-1$.

(b) Show that $|\mathbf{w}| = \det(\mathbf{v}_1, \dots, \mathbf{v}_{n-1}, \mathbf{n})$, where $\mathbf{n} = \mathbf{w}/|\mathbf{w}|$.

(c) Work out the vector $\mathbf{w}$ if $\mathbf{v}_a = \mathbf{e}_a$, for $a = 1, \dots, n-1$.

(d) Show that in three dimensions $\mathbf{w}$ can be written in terms of a cross product.

5. Solve the following system of linear equations

$$\begin{aligned} x + 2y + 3z &= 2 \\ 3x + 4y + 5z &= 4 \\ x + 3y + 4z &= 6 \end{aligned}$$

by

(a) calculating the matrix inverse

(b) Cramer's method

(c) row reduction on the augmented matrix.

If you had to write a computer program solving systems of linear equations (of arbitrary and possibly large size) which of the above methods would you base it on?

6.* On the vector space of polynomials $f : \mathbb{R} \to \mathbb{R}$ we define $\langle f, g \rangle = \int_{-\infty}^{\infty} dx\, e^{-x^2} f(x) g(x)$.

(a) Why does this define a scalar product?

(b) Consider the polynomials $p_0(x) = 1/n_0$, $p_1(x) = 2x/n_1$ and $p_2(x) = (4x^2 - 2)/n_2$, where the n_a are real numbers. Show that these polynomials are orthogonal under the above scalar product.

(c) Determine the numbers n_a such that the polynomials p_a are normalized to one, so $\langle p_a, p_a \rangle = 1$.

(Hint: You can use that $\int_{-\infty}^{\infty} dx\, x^n e^{-x^2} = 0$ for n odd (why?) and $\int_{-\infty}^{\infty} dx\, e^{-x^2} = \sqrt{\pi}$, $\int_{-\infty}^{\infty} dx\, x^2 e^{-x^2} = \sqrt{\pi}/2$, $\int_{-\infty}^{\infty} dx\, x^4 e^{-x^2} = 3\sqrt{\pi}/4$.)

7. The vector space V is equipped with a hermitian scalar product $\langle \cdot, \cdot \rangle$ and an ortho-normal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$.

(a) Show that non-zero and pairwise orthogonal vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are linearly independent.

(b) Show that the coordinates v_i of a vector $\mathbf{v}$, relative to the basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$, are given by $v_i = \langle \mathbf{e}_i, \mathbf{v} \rangle$.

(c) Show that the scalar product of two vectors $\mathbf{u}$, $\mathbf{v}$ can be written as $\langle \mathbf{u}, \mathbf{v} \rangle = \sum_{i=1}^{n} \langle \mathbf{u}, \mathbf{e}_i \rangle \langle \mathbf{e}_i, \mathbf{v} \rangle$.

(d) A second ortho-normal basis $\mathbf{e}'_1, \ldots, \mathbf{e}'_n$ is related to the first one by $\mathbf{e}'_j = \sum_i U_{ij} \mathbf{e}_i$, where U_{ij} are complex numbers. Show that $U_{ij} = \langle \mathbf{e}_i, \mathbf{e}'_j \rangle$ and that the matrix U with entries U_{ij} is unitary.

8. Consider $\mathbb{R}^n$ with the standard scalar product $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \mathbf{w}$ and orthogonal matrices R, that is, matrices satisfying $\langle R\mathbf{v}, R\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all vectors $\mathbf{v}$, $\mathbf{w}$.

(a) Show that orthogonal matrices can, alternatively, also be characterized by the equation $R^T R = \mathbf{1}$ and that $\det(R) = \pm 1$.

(b) Focus on $n = 2$ and show that two-dimensional orthogonal matrices can be written in the form

$$R(\varphi) = \begin{pmatrix} \cos\varphi & \mp \sin\varphi \\ \sin\varphi & \pm \cos\varphi \end{pmatrix} .$$

Which of these matrices correspond to two-dimensional rotations? What is the interpretation of the other matrices?

(c) For two-dimensional rotations, show that $R(\varphi_1) R(\varphi_2) = R(\varphi_1 + \varphi_2)$.

(d) The vectors $\mathbf{x} = (x, y)^T$ and $\mathbf{x}' = (x', y')^T$ are related by a rotation, so $\mathbf{x}' = R(\varphi)\mathbf{x}$. What is the relation between the two associated complex numbers $z = x + iy$ and $z' = x' + iy'$?

9.* (a) A three-dimensional rotation, R_3, around the z-axis can be constructed from the previous two-dimensional rotations by setting

$$R_3(\varphi) = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} ,$$

and analogously for three-dimensional rotations R_1 and R_2 around the x and y axis. Construct a general three-dimensional rotation by combining three such rotations, that is, work out $R = R_1(\alpha_1)R_2(-\alpha_2)R_3(\alpha_3)$ for three angles α_1, α_2 and α_3.

(b) Show that for small angles α_i the three-dimensional rotations from part (a) can be written as $R = \mathbf{1}_3 + \sum_{i=1}^{3} \alpha_i T_i + \cdots$, where the dots stand for terms of quadratic or higher order in the angles, and determine the (angle-independent) matrices T_1, T_2, T_3.

(c) The change of a vector $\mathbf{x}$ under a rotation R is given by $\delta\mathbf{x} = R\mathbf{x} - \mathbf{x}$. Show that for small-angle rotations this can be written as $\delta\mathbf{x} = \boldsymbol{\alpha} \times \mathbf{x} + \cdots$, where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)^T$ and the dots stand for quadratic or higher order terms in the angles.

10.* A bi-linear form on $\mathbb{R}^2$ is defined by $\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v}^T \eta \mathbf{w}$, where $\eta = \mathrm{diag}(-1, 1)$.

(a) Why is this bi-linear form not a scalar product?

(b) Consider the 2×2 matrices Λ which leave the above bi-linear form invariant, that is, which satisfy $\langle \Lambda\mathbf{v}, \Lambda\mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle$ for all vectors $\mathbf{v}$, $\mathbf{w}$. Show that these matrices can also be characterized by the equation $\Lambda^T \eta \Lambda = \eta$ and that $\det(\Lambda) = \pm 1$.

(c) Show that the matrices Λ with $\det(\Lambda) = 1$ and $\Lambda_{11} > 0$ can be written in the form

$$\Lambda(\xi) = \begin{pmatrix} \cosh \xi & \sinh \xi \\ \sinh \xi & \cosh \xi \end{pmatrix},$$

where ξ is a real parameter. Re-write Λ in terms of the parameter β, defined as $\beta = \tanh \xi$.

(d) Verify that $\Lambda(\xi_1)\Lambda(\xi_2) = \Lambda(\xi_1 + \xi_2)$. What does this rule imply for the parameter β, that is, if $\Lambda(\beta_1)\Lambda(\beta_2) = \Lambda(\beta)$, how does β depend on β_1 and β_2?

Additional computational problems

Computational methods, both numerical and symbolic, are of increasing importance in physics and symbolic computational tools have become significantly more powerful over the past decade or so. This is changing the way physicists work. Much as the introduction of the pocket calculator some 50 years ago has made by-hand numerical calculations unnecessary, modern systems such as Mathematica, can now take over standard symbolic calculations, such as algebraic manipulations or integration. This facilitates powerful checks of by-hand calculations but also allows for calculations which are virtually intractable with a pen-and-paper approach. The following problems present an opportunity to practice some of these methods in the context of topics from Linear Algebra. They are supplementary and voluntary but strongly recommended and hopefully a fun way to engage with symbolic computations early on. The problems are meant for realisation in Mathematica which can be downloaded from the university server. Mathematica is easy to use, has good built-in documentation and many high-level mathematical functions - you can start to experiment immediately.

C1) (Gram-Schmidt procedure)
Write a short module which takes a list of vectors, applies the Gram-Schmidt procedure and returns the orthogonalised version of the vectors. Use the code to verify your result from question 2.

C2) (Determinants)
Check your results from question 3b).

C3) (Scalar product on function spaces)
Check your results from question 6 b), c).

C4) (Rotations)

(a) Check your results from question 9.

(b) Think of the three angles α_i as time-dependent, so $\alpha_i = \alpha_i(t)$, so that the corresponding three-dimensional rotation also becomes time-dependent, $R = R(t)$. Then $R(t)$ may describe the linear transformation at time t between two coordinate system which rotate relative to one another. Compute the matrix $W = \dot{R}R^T$ (where the dot denotes the time derivative) and verify that it is anti-symmetric. Compute the angular velocity $\boldsymbol{\omega}$ (defined by $W_{ij} = \epsilon_{ikj}\omega_k$) and the angular speed $|\boldsymbol{\omega}|$ in terms of the angles α_i and their derivatives.