Vectors and Matrices, Problem Set 2

Matrices, linear maps and linear equations

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Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. For matrices A, B, C of compatible sizes and scalars α show that
(a) matrix multiplication is associate, so $A(BC) = (AB)C$,
(b) $(A+B)^T = A^T + B^T$, $(\alpha A)^T = \alpha A^T$ and $(AB)^T = B^T A^T$,
(c) $(A+B)^\dagger = A^\dagger + B^\dagger$, $(\alpha A)^\dagger = \alpha^* A^\dagger$ and $(AB)^\dagger = B^\dagger A^\dagger$,
(d) $(AB)^{-1} = B^{-1}A^{-1}$ and $(A^T)^{-1} = (A^{-1})^T$ provided A, B are invertible.

2. Using row operations, work out the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & -1 \\ 0 & 2 & -2 & 1 \\ 3 & 2 & 0 & -4 \\ 1 & -2 & a & 0 \end{pmatrix},$$

where a is a real parameter. (Keep in mind that the rank of A depends on the value of a so a case distinction is required.)

3. Using row operations, find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -2 \\ 1 & -3 & 0 \end{pmatrix}.$$

Check your result!

4. A linear map in $\mathbb{R}^2$ is given by the action of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}.$$

(a) Work out the matrix A' which represents this linear map in the basis

$$\mathbf{e}'_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{e}'_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(b) Find a 2×2 matrix P such that $A' = PAP^{-1}$.
(c) What is the interpretation of the matrix P?

5.* For a fixed unit vector $\mathbf{n}$ in $\mathbb{R}^3$ define the map $f : \mathbb{R}^3 \to \mathbb{R}^3$ by $f(\mathbf{v}) = \mathbf{v} - 2(\mathbf{n} \cdot \mathbf{v})\mathbf{n}$.
(a) Show that f is linear and that $f(f(\mathbf{v})) = \mathbf{v}$ for all vectors $\mathbf{v}$.
(b) Work out the matrix A describing f relative to the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of standard unit vectors and show that $A^2 = \mathbf{1}$.
(c) Work out the matrix $\hat{A}$ describing f relative to the basis $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{n}\}$, where $\mathbf{u}_a$, $a = 1, 2$, are two linearly independent vectors perpendicular to $\mathbf{n}$, so $\mathbf{u}_a \cdot \mathbf{n} = 0$.
(d) Discuss your results. What is the geometrical interpretation of f?

6. For a quadractic matrix A, the trace is defined as the sum of its diagonal elements, so $\mathrm{tr}(A) = \sum_i A_{ii}$.

(a) Show that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

(b) Show that $\mathrm{tr}(PAP^{-1}) = \mathrm{tr}(A)$. Discuss this result in relation to basis change of matrices.

(c)* What is the trace of the linear map f defined in the previous question?

7.* Consider the vector space consisting of all quadratic polynomials $p(x) = ax^2 + bx + c$, where a, b, c are arbitrary real coefficients. On this space a map is defined by

$$D(p) = x\frac{d^2p}{dx^2} + (1-x)\frac{dp}{dx} + 2p\,.$$

(a) Why is this map linear?

(b) Work out the matrix, A, which represents D for the standard monomial basis $\{1, x, x^2\}$.

(c) Find the kernel of the matrix A, that is, the vectors $\mathbf{v}$ satisfying $A\mathbf{v} = 0$. Which polynomials p correspond to these vectors?

(d) Show by explicitly applying D that the polynomials p found in part (c) satisfy $D(p) = 0$.

8. Analyze the linear system of equations

$$\begin{aligned} (2-\lambda)x + y + 2z &= 0 \\ x + (4-\lambda)y - z &= 0 \\ 2x - y + (2-\lambda)z &= 0\,, \end{aligned}$$

where λ is an arbitrary real parameter.

(a) What is the rank of the associated coefficient matrix, A, depending on the value of λ?

(b) Based on the rank result in (a) what is your expectation for the qualitative structure of the solution.

(c) Confirm your expectation by an explicit calculation.

9. The linear system

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + 4z &= \eta \\ x + 4y + 10z &= \eta^2 \end{aligned}$$

depends on the real parameter η.

(a) Show that the rank of the coefficient matrix is two. What does this imply for the qualitative structure of the solution?

(b) Explicitly solve the system for the cases where a solution exists.

10. Solve the linear system

$$\begin{aligned} 3x + 2y - z &= 10 \\ 5x - y - 4z &= 17 \\ x + 5y + \alpha z &= \beta\,, \end{aligned}$$

where α, β are arbitrary real parameters using row reduction on the augmented matrix.

11.* A semi-magic square is a 3×3 matrix of (rational) numbers such that all rows and columns sum up to the same total. A simple example is

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix},$$

where all rows and columns sum up to 6.

(a) Why do the semi-magic squares form a vector space?

(b) Show that the matrices

$$M_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

form a basis of the semi-magic squares. (Therefore, the dimension of this space is five and all semi-magic squares can be written as linear combinations of the above five matrices.)

12.* Consider an "internet" with n sites, labeled by $i = 1, \dots, n$. Each site i contains n_i links to some of the other sites and it is linked to by the pages $L_i \subset \{1, \dots, n\}$. The page rank x_i of each site i is defined by

$$x_i = \sum_{j \in L_i} \frac{x_j}{n_j} \; . \tag{1}$$

(a) Show that Eq. (1) can be written in matrix form as $A\mathbf{x} = 0$, where $\mathbf{x}$ is the vector with components x_i, and determine the components A_{ij} of the matrix A.

(b) Show a non-trivial solution (that is, a solution $\mathbf{x} \neq 0$) always exists.

(c) Analyze a simple example with $n = 4$ and the following structure of links:

$$\begin{array}{ccc} \mathbf{1} & \rightleftarrows & \mathbf{2} \\ \uparrow & \searrow & \downarrow \\ \mathbf{3} & \longleftarrow & \mathbf{4} \end{array}$$

Write down the page rank equations (1) for this case and solve them. Which page is ranked highest?

Additional computational problems

Computational methods, both numerical and symbolic, are of increasing importance in physics and symbolic computational tools have become significantly more powerful over the past decade or so. This is changing the way physicists work. Much as the introduction of the pocket calculator some 50 years ago has made by-hand numerical calculations unnecessary, modern systems such as Mathematica, can now take over standard symbolic calculations, such as algebraic manipulations or integration. This facilitates powerful checks of by-hand calculations but also allows for calculations which are virtually intractable with a pen-and-paper approach. The following problems present an opportunity to practice some of these methods in the context of topics from Linear Algebra. They are supplementary and voluntary but strongly recommended and hopefully a fun way to engage with symbolic computations early on. The problems are meant for realisation in Mathematica which can be downloaded from the university server. Mathematica is easy to use, has good built-in documentation and many high-level mathematical functions - you can start to experiment immediately.

C1) (Upper echelon form)
Write short piece of code which takes a matrix and brings it to upper echelon form. The output of the module should show all the steps of the calculation. Test the code on examples and use it to verify your result from question 2.

C2) (Systems of linear equations)
Check your results for the linear systems from questions 8, 9 and 10.

C3) (Graphs and page ranks)
Following on from question 12, write a short piece of code which takes a directed graph as an argument and which returns a list of the vertices ordered by page rank. (Note, Mathematica has convenient modules to construct graphs and extract their properties.) Use this code to check your result for the explicit graph from question 12 c) and also apply it to other examples.