Vectors and Matrices, Problem Set 1

Vectors, vector spaces and geometry

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Un-starred questions indicate standard problems students should be familiar with.
Starred questions refer to more ambitious problems which require a deeper understanding of the material.

1. Which of the following sets of vectors are linearly independent? For each linearly dependent set, identify a maximal subset of linearly independent vectors. Provide detailed reasoning in each case.

(a) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (b) $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

(c) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 1 \\ 6 \\ -1 \end{pmatrix}$ (d) $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ -3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -4 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -3 \\ 6 \\ -4 \\ 1 \end{pmatrix}$

2. Which of the following sets constitute sub vector spaces of the given vector space? Provide reasoning in each case.

(a) All three-dimensional vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ satisfying $x = y = 2z$.

(b) All two-dimensional vectors $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ satisfying $x^2 + y^2 = 1$.

(c) Within the vector space of real functions $f : \mathbb{R} \to \mathbb{R}$, the subset of even functions, that is, the functions satisfying $f(x) = f(-x)$.

3. Consider the vector space of 3×3 matrices A with real entries A_{ij}. What is the dimension of this vector space?

(a) Show that the subset of symmetric 3×3 matrices, that is, matrices satisfying $A_{ij} = A_{ji}$, forms a sub vector space. Write down an explicit basis for this sub vector space. What is its dimension?

(b) Do the same for the subset of 3×3 anti-symmetric matrices, that is, matrices satisfying $A_{ij} = -A_{ji}$.

(c) Generalize the results from (a) and (b) to $n \times n$ matrices with real entries.

4. Show that the vectors $\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ form a basis of $\mathbb{R}^3$. Write a general vector $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ as a linear combination of this basis. What are the coordinates of $\mathbf{v}$ relative to the basis $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$?

5.* Consider a real vector space V and two sub vector spaces U and W of V.

(a) Show that the intersection $U \cap W$ is a sub vector space of V.

(b) Show that $U + W$ (the set of all sums $\mathbf{u} + \mathbf{w}$ where $\mathbf{u} \in U$ and $\mathbf{w} \in W$) is a sub vector space of V.

(c) The dimensions of the above vector spaces are related by $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. Verify this formula for the specific example where $V = \mathbb{R}^3$, U is spanned by $\mathbf{u}_1 = \mathbf{i} + 2\mathbf{j}$, $\mathbf{u}_2 = \mathbf{k}$ and W is spanned by $\mathbf{w}_1 = \mathbf{j} + \mathbf{k}$, $\mathbf{w}_2 = -\mathbf{i} + 2\mathbf{j}$.

(If you are ambitious, try to prove the dimension formula in (c) in general. Start by writing down a basis for $U \cap W$ and complete this to a basis of U and W, respectively.)

6. For three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ prove the following relations:

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}\,,$

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}\,,$

(c) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})\,.$

7. For three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, prove the following:

(a) $\mathbf{a} \times \mathbf{b} = \mathbf{a} - \mathbf{b}$ implies that $\mathbf{a} = \mathbf{b}\,,$

(b) $\mathbf{c} = \lambda\mathbf{a} + \mu\mathbf{b}$ implies that $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0\,,$

(c) if $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$, this implies that $\mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = \pm|\mathbf{c}| \cdot |\mathbf{a} - \mathbf{b}|\,,$

(d) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}[\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})]\,.$

8. The three-dimensional vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ form the sides of a triangle.

(a) Show that $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}|$.

(b) Find the area of the triangle with vertices at $P = (1, 3, 2)$, $Q = (2, -1, 1)$ and $R = (-1, 2, 3)$.

9. (a) Prove that the three vectors $\mathbf{a} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{c} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ are coplanar.

(b) For $\mathbf{a} = \mathbf{i} + \mathbf{j} + \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$, find a vector which is coplanar with $\mathbf{a}$ and $\mathbf{b}$, but perpendicular to $\mathbf{a}$.

10.* The three-dimensional vectors $\mathbf{v}_i$, where $i = 1, 2, 3$ are not co-planar with triple product $V = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$. Define the *reciprocal vectors* $\mathbf{v}'_i$ by $\mathbf{v}'_1 = \frac{1}{V}\mathbf{v}_2 \times \mathbf{v}_3$, $\mathbf{v}'_2 = \frac{1}{V}\mathbf{v}_3 \times \mathbf{v}_1$ and $\mathbf{v}'_3 = \frac{1}{V}\mathbf{v}_1 \times \mathbf{v}_2$.

(a) Show that $\mathbf{v}_i \cdot \mathbf{v}'_j = \delta_{ij}$ and, hence, that the coordinates of a vector $\mathbf{w}$ relative to the basis $\mathbf{v}_i$ are given by $\mathbf{w} \cdot \mathbf{v}'_i$.

(b) Show that the triple product $V' = \mathbf{v}'_1 \cdot (\mathbf{v}'_2 \times \mathbf{v}'_3)$ of the reciprocal vectors equals $1/V$.

(c) Show that taking the reciprocal of the reciprocal leads back to the original vectors, that is, show that $\frac{1}{V'}\mathbf{v}'_2 \times \mathbf{v}'_3 = \mathbf{v}_1$, $\frac{1}{V'}\mathbf{v}'_3 \times \mathbf{v}'_1 = \mathbf{v}_2$ and $\frac{1}{V'}\mathbf{v}'_1 \times \mathbf{v}'_2 = \mathbf{v}_3$.

11. What is the shortest distance of $\mathbf{p} = (2, 3, 4)$ from the x-axis?

12. Write down the vector equation of the line

$$\frac{(x-2)}{4} = \frac{(y-1)}{3} = \frac{(z-5)}{2}$$

and find the minimum distance of this line from the origin.

13. Derive an expression for the shortest distance between the two lines $\mathbf{r}_i = \mathbf{q}_i + \lambda_i\mathbf{m}_i$, where $i = 1, 2$. Hence find the shortest distance between the lines

$$\frac{(x-2)}{2} = (y-3) = \frac{(z+1)}{2} \quad \text{and} \quad (x+2) = \frac{(y+1)}{2} = (z-1)\,.$$

14. Find the Cartesian equation for the plane passing through $P_1 = (2, -1, 1)$, $P_2 = (3, 2, -1)$ and $P_3 = (-1, 3, 2)$.

15. Find the Cartesian and vector description of the plane which contains the three position vectors $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{b} = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$, and $\mathbf{c} = 2\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$.

16. A line goes through the origin and the point $P = (1, 1, 1)$; a plane goes through points $A = (-1, 1, -2)$, $B = (1, 5, -5)$, $C = (0, 2, -3)$. Find the intersection point of the plane and the line.

Additional computational problems

Computational methods, both numerical and symbolic, are of increasing importance in physics and symbolic computational tools have become significantly more powerful over the past decade or so. This is changing the way physicists work. Much as the introduction of the pocket calculator some 50 years ago has made by-hand numerical calculations unnecessary, modern systems such as Mathematica, can now take over standard symbolic calculations, such as algebraic manipulations or integration. This facilitates powerful checks of by-hand calculations but also allows for calculations which are virtually intractable with a pen-and-paper approach. The following problems present an opportunity to practice some of these methods in the context of topics from Linear Algebra. They are supplementary and voluntary but strongly recommended and hopefully a fun way to engage with symbolic computations early on. The problems are meant for realisation in Mathematica which can be downloaded from the university server. Mathematica is easy to use, has good built-in documentation and many high-level mathematical functions - you can start to experiment immediately.

C1) (Linear independence computationally)
Check your results for question 1 (b), (c), (d), using Mathematica.

C2) (Cross and dot product identities)
Prove the relations from question 6 (a), (b), (c) computationally.

C3) (Planes)
Write a short piece of code which finds the Cartesian and vector form of a plane in $\mathbb{R}^3$ which contains three given points p_1, p_2 and p_3. Use this code to check your results for questions 14 and 15.

C4) (Perceptron)
Read the short application about perceptrons in the lecture notes. Realise such a perceptron in Mathematica and train it as explained in the lecture notes, that is, with a set of points in $\mathbb{R}^2$ above and below a given line. Check that the trained perceptron successfully detects whether given points are above or below. (Mathematica has built-in modules to construct neural networks so this is much easier than it sounds.)