Mathematical Methods MT2017: Problems 2

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1.

(i) Show that the quadratic form $4x^2 + 2y^2 + 2z^2 - 2xy + 2yz - 2zx$ can be written as $\vec{x}^{\mathrm{T}} V \vec{x}$ where V is a symmetric matrix. Find the eigenvalues of V. Explain why, by rotating the axes, the quadratic form may be reduced to the simple expression $\lambda x'^2 + \mu y'^2 + \nu z'^2$; what are λ, μ, ν?

(ii) The components of the current density vector $\vec{j}$ in a conductor are proportional to the components of the applied electric field $\vec{E}$ in simple (isotropic) cases: $\vec{j} = \sigma \vec{E}$. In crystals, however, the relation may be more complicated, though still *linear*, namely of the form $j_a = \sum_{b=1}^{3} \sigma_{ab} E_b$, where σ_{ab} for the entries in a real symmetric 3×3 matrix and the index a runs from 1 to 3. In a particular case, the quantities σ_{ab} are given (in certain units) by $\begin{pmatrix} 4 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$. Explain why by a rotation of the axes the relation between $\vec{j}$ and $\vec{E}$ can be reduced to $j'_1 = \tilde{\sigma}_1 E'_1$, $j'_2 = \tilde{\sigma}_2 E'_2$, $j'_3 = \tilde{\sigma}_3 E'_3$ and find $\tilde{\sigma}_1$, $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$.

2. This question assumes that you are familiar with some basic notions of probability: refer to Steve Biller's Probability & Statistics course or Appendix B onwards of the Maths Methods notes if necessary.

The probability density function (PDF) of a Gaussian (or Normal) distribution centred on the origin in n-dimensional space is given by

$$\mathrm{p}(\mathbf{x}) = B \exp\left[-\tfrac{1}{2}\mathbf{x}^T \Lambda \mathbf{x}\right], \tag{Q2.1}$$

where $\mathbf{x} = (x_1, ..., x_n)$, Λ is an $n \times n$ matrix, known as the "precision matrix", and B is a normalising factor.

(i) By considering the conditions needed for this $\mathrm{p}(\mathbf{x})$ to be a bona fide PDF, what properties may we assume for the precision matrix Λ? How is Λ related to the covariance matrix V of the distribution?

(ii) Hence show that the normalisation condition $\int \mathrm{p}(\mathbf{x})\,\mathrm{d}^n\mathbf{x} = 1$ requires that $B = \sqrt{\det\Lambda/(2\pi)^n}$. [Recall that $\int_{-\infty}^{\infty} \mathrm{e}^{-\frac{1}{2}x^2}\,\mathrm{d}x = \sqrt{2\pi}$.]

(iii) By diagonalising the quadratic form $\mathbf{x}^T\Lambda\mathbf{x}$, or otherwise, sketch $\mathrm{p}(\mathbf{x})$ for the case $\Lambda = \frac{1}{2}\begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix}$.

(iv) We may take the elements x_1 and x_2 of the vector $\mathbf{x}$ in part (iii) as random variables. Stating the conditions needed for an arbitrary pair of random variables, A and B to be independent, explain whether x_1 and x_2 are independent.

(v) How are the mean and variance of a continuous random variable defined? Calculate the mean and variance of x_1.

(vi) Consider a change of variables from (x_1, x_2) to (r, ϕ) defined implicitly by $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. Find the PDF $\mathrm{p}(r, \phi)$. Are the random variables r and ϕ independent? Are they correlated?

3. While we're briefly distracted by probability theory, suppose that X and Y are independent continuous random variables having PDFs $f_X(x)$ and $f_Y(Y)$. Show that the PDF of $Z = X + Y$ is

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z - x)\,\mathrm{d}x, \tag{Q3.1}$$

and obtain an expression for the PDF of $U = XY$.

[This expression for $f_Z(z)$ is intuitively obvious, particularly if you think of the corresponding result for the probability mass functions of discrete random variables. But you can show it for continuous random variables by transforming from (X, Y) to (Z, W) for some W that you then marginalise.]

4. Let Λ be a symmetric $n \times n$ matrix. Show that $\mathbf{x}^T \Lambda \mathbf{x} = \operatorname{tr}(\Lambda \mathbf{x}\mathbf{x}^T)$.

Show further that, if Λ_a and Λ_b are symmetric $n \times n$ matrices, then

$$\begin{aligned}&\operatorname{tr}\left[\Lambda_a(\mathbf{x}-\bar{\mathbf{x}}_a)(\mathbf{x}-\bar{\mathbf{x}}_a)^T + \Lambda_b(\mathbf{x}-\bar{\mathbf{x}}_b)(\mathbf{x}-\bar{\mathbf{x}}_b)^T\right] \\ &\quad = \operatorname{tr}\left[\Lambda_{ab}(\mathbf{x}-\bar{\mathbf{x}}_{ab})(\mathbf{x}-\bar{\mathbf{x}}_{ab})^T + \Lambda_a\Lambda_{ab}^{-1}\Lambda_b(\bar{\mathbf{x}}_a-\bar{\mathbf{x}}_b)(\bar{\mathbf{x}}_a-\bar{\mathbf{x}}_b)^T\right],\end{aligned} \tag{Q4.1}$$

where $\Lambda_{ab} \equiv \Lambda_a + \Lambda_b$ and $\bar{\mathbf{x}}_{ab} \equiv \Lambda_{ab}^{-1}(\Lambda_a\bar{\mathbf{x}}_a + \Lambda_b\bar{\mathbf{x}}_b)$. This is the n-dimensional generalisation of "completing the square".

(i) Consider a pair of Gaussians $i = 1, 2$,

$$g_i(\mathbf{x}) = \frac{\sqrt{\det \Lambda_i}}{(2\pi)^{n/2}} \exp\left[-\tfrac{1}{2}(\mathbf{x}-\bar{\mathbf{x}}_i)^T\Lambda_i(\mathbf{x}-\bar{\mathbf{x}}_i)\right] \tag{Q4.2}$$

having means $\bar{\mathbf{x}}_i$ and precisions Λ_i. Show that their product $g_1(\mathbf{x})g_2(\mathbf{x})$ is another (unnormalized) Gaussian with precision $\Lambda = \Lambda_1 + \Lambda_2$ and mean $\Lambda^{-1}(\Lambda_1\bar{\mathbf{x}}_1 + \Lambda_2\bar{\mathbf{x}}_2)$.

(ii) Show that the convolution of two Gaussians,

$$h(\mathbf{x}) = \int g_1(\mathbf{x}')g_2(\mathbf{x}-\mathbf{x}')\mathrm{d}\mathbf{x}', \tag{Q4.3}$$

is another Gaussian. What is its mean and variance?

5. Show by direct expansion of the determinant that

$$\det(I + \epsilon A) = 1 + \epsilon \operatorname{tr} A + O(\epsilon^2) \tag{Q5.1}$$

and hence that

$$\det(\exp A) = \exp(\operatorname{tr} A). \tag{Q5.2}$$

6. (i) Let A be a matrix having elements A_{ij}. Show that

$$\frac{\partial \det A}{\partial A_{ij}} = (\operatorname{adj} A)_{ji}, \tag{Q6.1}$$

where $\operatorname{adj} A$ is the adjugate matrix of A.

(ii) Suppose that the elements of A depend on a parameter α. Show that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\det A = (\det A)\operatorname{tr}\left(A^{-1}\frac{\mathrm{d}A}{\mathrm{d}\alpha}\right). \tag{Q6.2}$$

(iii) Consider a fluid moving in n-dimensional space. Its motion defines a *flow*: a smooth mapping $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ of the space onto itself for which $\mathbf{x}(\mathbf{X}, 0) = \mathbf{X}$. Show that the Jacobian matrix of this mapping, $J_{ij} = \partial x_i/\partial X_j$, satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\det J = (\det J)\nabla\cdot\mathbf{u}, \tag{Q6.3}$$

where $\mathbf{u}$ is the velocity vector having elements $u_i = \mathrm{d}x_i/\mathrm{d}t$. State clearly any assumptions you make about the arguments of J and $\mathbf{u}$.

(iv) Suppose that some dye is added to the fluid and is advected along with it [that is, the dye at position $\mathbf{x}$ moves with velocity $\mathbf{u}(\mathbf{x}, t)$]. Prove *Reynold's transport theorem*: for any function $G(\mathbf{x}, t)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} G\,\mathrm{d}^n\mathbf{x} = \int_{V(t)}\left(\frac{\mathrm{D}G}{\mathrm{D}t} + G\nabla\cdot\mathbf{u}\right)\mathrm{d}^n\mathbf{x}, \tag{Q6.4}$$

where $V(t)$ denotes the volume of space that the dye occupies at time t and the total derivative $\frac{\mathrm{D}}{\mathrm{D}t} \equiv \frac{\partial}{\partial t} + (\mathbf{u}\cdot\nabla)$. Give a geometrical interpretation of this result.