Mathematical Methods MT2017: Problems 1

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1. Find real values α and β such that the complex vectors $\mathbf{u} = \alpha\begin{pmatrix}1+\mathrm{i}\\1-\mathrm{i}\end{pmatrix}$ and $\mathbf{v} = \beta\begin{pmatrix}1-\mathrm{i}\\1+\mathrm{i}\end{pmatrix}$ are normalised. What is the value of the scalar product $\mathbf{u}^\dagger\cdot\mathbf{v}$? Prove that $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. Are there any further linearly independent two-dimensional complex vectors? If so, find the necessary vectors to make an orthogonal basis. Express the vector $\begin{pmatrix}1\\\mathrm{i}\end{pmatrix}$ as a linear combination of the basis vectors.

2. Use the Gram–Schmidt procedure to construct an orthonormal set of vectors from the following: $\vec{x}_1 = (0,0,1,1)^{\mathrm{T}}$, $\vec{x}_2 = (1,0,-1,0)^{\mathrm{T}}$, $\vec{x}_3 = (1,2,0,2)^{\mathrm{T}}$, $\vec{x}_4 = (2,1,1,1)^{\mathrm{T}}$.

3. Let $S(\mathbf{a},\mathbf{b}) = \mathbf{a}^\dagger M\mathbf{b}$, where M is a matrix. What conditions do we need to impose on M if $S(\mathbf{a},\mathbf{b})$ is to define a scalar product between the vectors $\mathbf{a}$ and $\mathbf{b}$?

4. Prove the **triangle inequality**: given an inner product $\langle\cdot,\cdot\rangle$ on a vector space $\mathcal{V}$, show that the norm $\|\mathbf{a}\|$ defined by $\|\mathbf{a}\|\equiv\sqrt{\langle\mathbf{a},\mathbf{a}\rangle}$ satisfies $\|\mathbf{v}+\mathbf{w}\|\le\|\mathbf{v}\|+\|\mathbf{w}\|$ for any $\mathbf{v}$, $\mathbf{w}\in\mathcal{V}$.

5. Let A be an operator on a complex vector space $\mathcal{V}$ and let $|v_1\rangle$ and $|v_2\rangle$ be any two vectors from $\mathcal{V}$. Verify that

$$4\langle v_2|\,A\,|v_1\rangle = \langle v_+|\,A\,|v_+\rangle - \langle v_-|\,A\,|v_-\rangle + \langle v'_+|\,A\,|v'_+\rangle\mathrm{i} - \langle v'_-|\,A\,|v'_-\rangle\mathrm{i}, \tag{Q5.1}$$

where $|v_\pm\rangle\equiv|v_1\rangle\pm|v_2\rangle$ and $|v'_\pm\rangle\equiv|v_1\rangle\pm\mathrm{i}\,|v_2\rangle$. Use this result to show that

(i) if $\langle v|\,A\,|v\rangle = 0$ for all vectors $|v\rangle\in\mathcal{V}$ then $A=0$;

(ii) A is Hermitian if and only if $\langle v|\,A\,|v\rangle$ is real for all $|v\rangle\in\mathcal{V}$. (Hint: if $\langle v|\,A\,|v\rangle$ is real then $\langle v|\,A\,|v\rangle - \langle v|\,A\,|v\rangle^\star = 0$.)

6. Which of these matrices represents a rotation?

$$\begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} \frac{1}{4} & \frac{3}{4} & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & \sqrt{\frac{3}{8}} \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & -\frac{1}{2} \end{pmatrix}. \tag{Q6.1}$$

Find the angle and axis of the rotation. What does the other matrix represent?

7. Show that if $[A,B]=0$ then

$$(A+B)^n = \sum_{i=0}^{n}\binom{n}{i}A^iB^{n-i}, \tag{Q7.1}$$

identifying clearly where you make use of the fact that $[A,B]=0$. Use this result to show that if $[A,B]=0$ then $\exp A\exp B = \exp(A+B)$. Give an example of A,B for which $[A,B]\neq 0$ and $\exp A\exp B\neq\exp(A+B)$.

8. Consider the matrices $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$, $\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Which of the matrices are symmetric? Which are Hermitian? By calculating the commutators of these matrices, show that they can be written as $[\sigma^a, \sigma^b] = 2\mathrm{i}\epsilon_{abc}\sigma^c$, where ϵ_{abc} is the alternating tensor and on the right-hand side the summation convention is employed (i.e., the index c is summed over). Write $\exp(\mathrm{i}\alpha\sigma^y)$ (α is a real number) as a 2×2 matrix. What does it represent? Show that $\exp(\mathrm{i}\alpha\sigma^y)$ is unitary without writing it explicitly as a 2×2 matrix.

9.

(a) Find the eigenvalues of the Pauli matrix $\sigma^y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$. Normalise the two corresponding eigenvectors, $\vec{u}_1$, $\vec{u}_2$, so that $\vec{u}_1^\dagger \cdot \vec{u}_1 = \vec{u}_2^\dagger \cdot \vec{u}_2 = 1$. Check that $\vec{u}_1^\dagger \cdot \vec{u}_2 = 0$. Form the matrix $U = (\, \vec{u}_1 \quad \vec{u}_2 \,)$ and verify that $U^\dagger U = I$. Evaluate $U^\dagger \sigma^y U$. What have you learned from this calculation?

(b) A general 2-component complex vector $\vec{v} = (c_1, c_2)^{\mathrm{T}}$ is expanded as a linear combination of the eigenvectors $\vec{u}_1$ and $\vec{u}_2$ via

$$\vec{v} = \alpha\vec{u}_1 + \beta\vec{u}_2, \tag{Q9.1}$$

where α and β are complex numbers. Determine α and β in terms of c_1 and c_2 in two ways: (i) by equating corresponding components of (Q9.1), (ii) by showing that $\alpha = \vec{u}_1^\dagger \cdot \vec{v}$, $\beta = \vec{u}_2^\dagger \cdot \vec{v}$ and evaluating these products.

10. Construct a real symmetric matrix whose eigenvalues are 2, 1 and -2, and whose corresponding normalised eigenvectors are $\frac{1}{\sqrt{2}}(0,1,1)^{\mathrm{T}}$, $\frac{1}{\sqrt{2}}(0,1,-1)^{\mathrm{T}}$ and $(1,0,0)^{\mathrm{T}}$.

11. Find the eigenvalues and eigenvectors of the matrix $F = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$. Hence, proving the validity of the method you use, find the values of the elements of the matrix F^n, where n is a positive integer.

12. By finding the eigenvectors of the Hermitian matrix $H = \begin{pmatrix} 10 & 3\mathrm{i} \\ -3\mathrm{i} & 2 \end{pmatrix}$ construct a unitary matrix U such that $U^\dagger H U = D$, where D is a real diagonal matrix.

13. Which of the following matrices have a complete set of eigenvectors in common?

$$A = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}, \quad C = \begin{pmatrix} -9 & -10 \\ -10 & 5 \end{pmatrix}, \quad D = \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}. \tag{Q13.1}$$

Construct the common set of eigenvectors where possible.